Calculating the area and centroid of a polygon in 2d

Let $\{(x_i, y_i)\}_{i=0}^{N-1} \subset \mathbb{R}^2$ be a closed polygon in the plane, and let the vertices be ordered counter clockwise. Then it is well-known that the polygon encloses the area

$$A = \frac{1}{2} \sum_{i=0}^{N-1} (x_i\, y_{i+1} - x_{i+1}\, y_i)\,,$$

and its centroid is given by

$$\frac{1}{6\,A} \left(\sum_{i=0}^{N-1} (x_i + x_{i+1})\,(x_i\, y_{i+1} - x_{i+1}\, y_i), \sum_{i=0}^{N-1} (y_i + y_{i+1})\,(x_i\, y_{i+1} - x_{i+1}\, y_i) \right)^T \in \mathbb{R}^2\,;$$

see e.g. `paulbourke.net/geometry/polygonmesh`.

Calculating the volume and centroid of a polyhedron in 3d

Similar formulas exist for the enclosed volume and centroid of a polyhedron P in $\mathbb{R}^3$, but these appear to be less well-known. In the following we assume without loss of generality that the boundary of the polyhedron is given by a union of triangles. (More general facets can easily be subdivided into triangles.) We stress that P need not be convex.

Let A_i, $i = 0, \ldots, N-1$, be the N triangular faces of the polyhedron, with vertices (a_i, b_i, c_i), which are assumed to be ordered counter clockwise on A_i. This means that we can define the outer unit normal n to P on each A_i as $n_i = \hat{n}_i/|\hat{n}_i|$, where $\hat{n}_i = (b_i - a_i) \otimes (c_i - a_i)$. Then the volume of P is given by

$$V = \int_P 1 = \frac{1}{3} \int_{\partial P} x \cdot n = \frac{1}{3} \sum_{i=0}^{N-1} \int_{A_i} a_i \cdot n_i = \frac{1}{6} \sum_{i=0}^{N-1} a_i \cdot \hat{n}_i\,,$$

where we have used the divergence theorem, the fact that $x \cdot n_i$ is constant on each A_i, and the fact that the area of A_i is given by $\frac{1}{2}\,|\hat{n}_i|$.

Let $c \in \mathbb{R}^3$ denote the centroid of P, i.e. $c = \frac{1}{V} \int_P x$. Applying the divergence theorem once again, and on denoting the standard basis in $\mathbb{R}^3$ by $\{e_1, e_2, e_3\}$, we obtain for the three coordinates of the centroid that

$$c \cdot e_d = \frac{1}{V} \int_{\partial P} \frac{1}{2}\,(x \cdot e_d)^2\,(n \cdot e_d) = \frac{1}{2V} \sum_{i=0}^{N-1} \int_{A_i} (x \cdot e_d)^2\,(n_i \cdot e_d)\,, \qquad d = 1, 2, 3\,.$$

It remains to compute that

$$\begin{aligned}\int_{A_i} (x \cdot e_d)^2\,(n_i \cdot e_d) &= \frac{1}{6}\,\hat{n}_i \cdot e_d \left([\tfrac{1}{2}\,(a_i + b_i) \cdot e_d]^2 + [\tfrac{1}{2}\,(b_i + c_i) \cdot e_d]^2 + [\tfrac{1}{2}\,(c_i + a_i) \cdot e_d]^2 \right) \\ &= \frac{1}{24}\,\hat{n}_i \cdot e_d \left([(a_i + b_i) \cdot e_d]^2 + [(b_i + c_i) \cdot e_d]^2 + [(c_i + a_i) \cdot e_d]^2 \right),\end{aligned}$$

where we have observed that the integrand is a quadratic function on A_i, so that the standard midpoint sampling quadrature formula for triangles yields the integral exactly, see e.g. [1].

References

[1] A. H. Stroud, *Approximate calculation of multiple integrals*, Prentice-Hall Inc., Englewood Cliffs, N. J., 1971.