On the Perimeter of an Ellipse

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Computing accurate approximations to the perimeter of an ellipse is a favourite problem of amateur mathematicians, even attracting luminaries such as Ramanujan [1, 2, 3]. As is well known, the perimeter, \( P \), of an ellipse with semimajor axis \( a \) and semiminor axis \( b \) can be expressed exactly as a complete elliptic integral of the second kind, which can also be written as a Gaussian hypergeometric function,

\[
P = 4a E \left( 1 - \frac{b^2}{a^2} \right) = 2\pi a \, _2F_1 \left( \frac{1}{2}, -\frac{1}{2}; 1; 1 - \frac{b^2}{a^2} \right)
\]

(1)

What is less well known is that the various exact forms attributed to Maclaurin, Gauss-Kummer, and Euler, are related via quadratic transformation formulae for hypergeometric functions. In this way we obtain additional identities, including a particularly elegant formula, symmetric in \( a \) and \( b \),

\[
P = 2\pi \sqrt{ab} \, P_{\frac{1}{2}} \left( \frac{a^2 + b^2}{2ab} \right)
\]

(2)

where \( P_{\nu}(z) \) is a Legendre function.

Approximate formulae can be obtained by truncating the series representations of exact formulas. For example, Kepler used the geometric mean, \( P \approx 2\pi \sqrt{ab} \). In this paper, we examine the properties of a number of approximate formulas, using series methods, polynomial interpolation, rational polynomial approximants, and minimax methods.
**Cartesian Equation**

The Cartesian equation for an ellipse with centre at \((0, 0)\), semimajor axis \(a\), and semiminor axis \(b\) reads

\[
  E(x, y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1;
\]

Introducing the parameter \(\phi\) into the Cartesian coordinates, as \((x = a \sin(\phi), y = b \cos(\phi))\), one verifies that the ellipse equation is satisfied.

\[
  \text{In[2]} := \text{Simplify}[E(a \sin(\phi), b \cos(\phi))]
\]

\[
  \text{Out[2]} = \text{True}
\]

**Arclength**

In general, the parametric arclength is defined by

\[
  L = \int_{\phi_1}^{\phi_2} \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2} \, d\phi
\]  \hspace{1cm} (3)

The arclength of an ellipse as a function of the parameter \(\phi\) is an (incomplete) elliptic integral of the second kind.

\[
  \text{In[3]} := \text{L}(\phi) = \text{With}[\{x = a \sin(\phi), y = b \cos(\phi)\},
  \text{Simplify}\left[\int \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2} \, d\phi, a > b > 0 \land 0 < \phi < \frac{\pi}{2}\}\right]
\]

\[
  \text{Out[3]} = a E\left(\phi \mid 1 - \frac{b^2}{a^2}\right)
\]

Since,

\[
  \text{In[4]} := L(0) = 0
\]

\[
  \text{Out[4]} = \text{True}
\]

the arclength of the ellipse is

\[
  L(\phi) = a E(\phi \mid e^2)
\]  \hspace{1cm} (4)

where the eccentricity, \(e\), is defined by

\[
  \text{In[5]} := e(a, b) = \sqrt{1 - \frac{b^2}{a^2}}
\]

**Perimeter**

Since the parameter ranges over \(0 \leq \phi \leq \pi/2\) for one quarter of the ellipse, the perimeter of the ellipse is

\[
  \text{In[6]} := P_r(a, b) = 4 \int_0^{\pi/2} L(\phi)
\]

\[
  \text{Out[6]} = 4a E\left(1 - \frac{b^2}{a^2}\right)
\]
That is $\mathcal{P} = 4a \, E(e^2)$ where $E(m)$ is the complete elliptic integral of the second kind.

### Alternative Expressions for the Perimeter

The above expression for the perimeter of the ellipse is *unsymmetrical* with respect to the parameters $a$ and $b$. This is “unphysical” in that both parameters, being lengths of the (major and minor) axes, should be on the same footing. We can expect that a symmetric formula, when truncated, will more accurately approximate the perimeter for both $a \geq b$ and $a \leq b$.

Noting that the complete elliptic integral is a gaussian hypergeometric function,

$$
\ln[7] := 2F_1 \left( \frac{1}{2}, -\frac{1}{2}; 1; z \right)
$$

$$
\text{Out}[7] := \frac{2 \, E(z)}{\pi}
$$

one obtains Maclaurin's 1742 formula (see [2])

$$
\ln[8] := 2 \pi a \, 2F_1 \left( \frac{1}{2}, -\frac{1}{2}; 1; e(a, b)^2 \right)
$$

$$
\text{Out}[8] := \text{True}
$$

Equivalent alternative expressions for the perimeter of the ellipse can be obtained from quadratic transformation formulæ for gaussian hypergeometric functions. For example, using functions.wolfram.com/07.23.17.0106.01,

$$
\ln[9] := \text{Simplify} \left[ F_1 (\alpha, \beta; 2 \, \beta; z) = \frac{2F_1 \left( \alpha, \alpha - \beta + \frac{1}{2}; \beta + \frac{1}{2}; \left( \frac{1 - \sqrt{1 - z}}{\sqrt{1 - z} + 1} \right)^2 \right)}{\left( \frac{1}{2} \left( \sqrt{1 - z} + 1 \right) \right)^{2\beta}} \right]
$$

$$
\{ \beta \to \frac{1}{2}, \alpha \to -\frac{1}{2}, z \to e(a, b)^2 \}, \ a > b > 0
$$

$$
\text{Out}[9] := 4 \, a \, E \left( 1 - \frac{b^2}{a^2} \right) = (a + b) \, \pi \, 2F_1 \left( -\frac{1}{2}, -\frac{1}{2}; 1; \frac{(a - b)^2}{(a + b)^2} \right)
$$

and noting that

$$
\ln[10] := \frac{(a - b)^2}{(a + b)^2} = 1 - \frac{4 \, a \, b}{(a + b)^2} \quad // \quad \text{Simplify}
$$

$$
\text{Out}[10] := \text{True}
$$

one obtains the following symmetric formula

$$
\ln[11] := \mathcal{P}_s(a, b) = \pi \, (a + b) \, 2F_1 \left( -\frac{1}{2}, -\frac{1}{2}; 1; 1 - \frac{4 \, a \, b}{(a + b)^2} \right);
$$

first obtained by Ivory (1796), but known as the Gauss-Kummer series (see [2]).

Introducing the homogenous symmetric parameter $h = \frac{(a-b)^2}{(a+b)^2} = 1 - \frac{4 \, a \, b}{(a+b)^2}$, one has (c.f. mathworld.wolfram.com/Ellipse.html),

$$
\ln[12] := \pi \, (a + b) \, 2F_1 \left( -\frac{1}{2}, -\frac{1}{2}; 1; h \right) \quad // \quad \text{FunctionExpand} \quad // \quad \text{Simplify}
$$

$$
\text{Out}[12] := 2(a + b) \, (2 \, E(h) + (h - 1) \, K(h))
$$

Explicitly, the Gauss-Kummer series reads

$$
\ln[13] := \mathcal{P}_s(a, b) = \text{FullSimplify}[\mathcal{P}_s(a, b) \quad // \quad \text{FunctionExpand}, \ a > b > 0]
$$

$$
\text{Out}[13] := 4 \, (a + b) \, E \left( 1 - \frac{4 \, a \, b}{(a + b)^2} \right) - \frac{8 \, a \, b \, K \left( 1 - \frac{4 \, a \, b}{(a + b)^2} \right)}{a + b}
$$
Instead, using functions.wolfram.com/07.23.17.0103.01, one obtains Euler's 1773 formula (see also [2]):

\[
\ln[14] = \, _2F_1(\alpha, \beta; 2\beta; z) = \frac{\, _2F_1\left(\frac{\alpha + 1}{2}; \beta + \frac{1}{2}; \frac{1}{1 - z^2}\right)}{(1 - z)^{\beta - \frac{1}{2}}}.
\]

This perimeter can be expressed in terms of Legendre functions (see sections 8.13 and 15.4 of [4]). For example, using 15.4.15 of [4] one obtains an elegant and simple symmetric formula

\[
\ln[19] = \, P_2(a, b) = \frac{16 \, a^2 \, b^2 \, \pi \, \frac{2 \, F_1(\alpha, \beta; \gamma; z_1)}{\left(\frac{1}{2}; \frac{3}{4}, 1; 1 - \frac{4 \, a \, b}{(a + b)^2}\right)}}{(a + b)^3}
\]

The perimeter can also be expressed in terms of Legendre functions (see sections 8.13 and 15.4 of [4]). For example, using 15.4.15 of [4] one obtains an elegant and simple symmetric formula

\[
\ln[19] = \, \Gamma(a - b + 1) (1 - x)^{-\frac{1}{2}} = \frac{\, b^2}{a^2 \, \pi \, \frac{2 \, F_1(\alpha, \beta; \gamma; z_1)}{\left(\frac{1}{2}; \frac{3}{4}, 1; 1 - \frac{4 \, a \, b}{(a + b)^2}\right)}}{(a + b)^3}
\]

This form can be used to prove that the perimeter of an ellipse is a homogenous mean (c.f. [5]), extending the arithmetic-geometric mean (AGM) already used as a tool for computing elliptic integrals [6].

Using functions.wolfram.com/07.07.26.0001.01, this gives yet another formula involving complete elliptic integrals.

\[
\ln[20] = \, P_2(a, b) = \frac{16 \, a^2 \, b^2 \, \pi \, \frac{2 \, F_1(\alpha, \beta; \gamma; z_1)}{\left(\frac{1}{2}; \frac{3}{4}, 1; 1 - \frac{4 \, a \, b}{(a + b)^2}\right)}}{(a + b)^3}
\]

\[
\ln[21] = \, 4 \, \sqrt{a \, b} \, \pi \, F_1\left(\frac{a^2 + b^2}{2 \, a \, b}, 1; \frac{1}{2}; \frac{a^2 + b^2}{2 \, a \, b}\right)
\]

\[
\ln[22] = \, 4 \, \sqrt{a \, b} \, \frac{\, a \, b}{2 \, a \, b} - K\left(\frac{b^2}{a^2 + b^2}\right)
\]

\[
\ln[23] = \, 4 \, \sqrt{a \, b} \, \left(\frac{a^2 + b^2}{2 \, a \, b} - K\left(\frac{b^2}{a^2 + b^2}\right)\right)
\]

\[
\ln[24] = \, 4 \, \sqrt{a \, b} \, \left(\frac{a^2 + b^2}{2 \, a \, b} - K\left(\frac{b^2}{a^2 + b^2}\right)\right)
\]
Comparisons

Here we compare the seven formulas obtained above for $b = 2a$.

$$\text{In}[21] = \text{Simplify}[[P_1(a, 2a), P_2(a, 2a), P_3(a, 2a),}$$
$$P_4(a, 2a), P_5(a, 2a), P_6(a, 2a), P_7(a, 2a)], a > 0]$$

$$\text{Out}[21] = \begin{array}{l}
4 a E(-3), 3 a \pi \frac{F_1}{\left(-\frac{1}{2}, -\frac{1}{2}; 1; \frac{1}{9}\right)}, \frac{4}{3} a\left(9 E\left(\frac{1}{9}\right) - 4 K\left(\frac{1}{9}\right)\right), \\
\sqrt{10} a \pi \frac{F_1}{\left(\frac{1}{4}, -\frac{1}{4}; 1; \frac{9}{25}\right)}, 64 a \pi \frac{F_1}{\left(\frac{3}{2}, \frac{3}{2}; 1; \frac{1}{9}\right)}, \\
2 \sqrt{2} a \pi P_{\frac{5}{4}}, 4 \sqrt{2} a\left(2 E\left(-\frac{1}{8}\right) - K\left(-\frac{1}{8}\right)\right) \\
\end{array}$$

$$\text{In}[22] = N[%]$$

$$\text{Out}[22] = (9.688448221 a, 9.688448221 a, 9.688448221 a, 9.688448221 a, 9.688448221 a)$$

$$\text{In}[23] = \text{Equal} @@ %$$

$$\text{Out}[23] = \text{True}$$

and for $b = a / 3$.

$$\text{In}[24] = \text{Simplify}[$$
$$[P_1(a, \frac{a}{3}), P_2(a, \frac{a}{3}), P_3(a, \frac{a}{3}), P_4(a, \frac{a}{3}), P_5(a, \frac{a}{3}), P_6(a, \frac{a}{3}), P_7(a, \frac{a}{3})], a > 0]$$

$$\text{Out}[24] = \begin{array}{l}
4 a E\left(\frac{8}{9}\right), 3 a \pi \frac{F_1}{\left(-\frac{1}{2}, -\frac{1}{2}; 1; \frac{1}{4}\right)}, \\
\frac{2}{3} a\left(8 E\left(\frac{1}{4}\right) - 3 K\left(\frac{1}{4}\right)\right), \frac{2}{3} \sqrt{5} a \pi \frac{F_1}{\left(\frac{1}{4}, -\frac{1}{4}; 1; \frac{16}{25}\right)}, \\
\frac{3}{4} a \pi \frac{F_1}{\left(\frac{3}{2}, \frac{3}{2}; 1; \frac{1}{4}\right)}, \frac{2 a \pi P_{\frac{5}{4}}}{\sqrt{3}}, a\left(8 E\left(-\frac{1}{8}\right) - 4 K\left(-\frac{1}{8}\right)\right) \\
\end{array}$$

$$\text{In}[25] = N[%]$$

$$\text{Out}[25] = (4.454964407 a, 4.454964407 a, 4.454964407 a, 4.454964407 a, 4.454964407 a, 4.454964407 a)$$

$$\text{In}[26] = \text{Equal} @@ %$$

$$\text{Out}[26] = \text{True}$$

### Numerical Approximation

At www.ebyte.it/library/docs/math05a/EllipsePerimeterApprox05.html [1] one is encouraged to search for “an efficient formula using only the four algebraic operations (if possible, avoiding even square-root) with a maximum error below 10 parts per million. If would be also nice if such a formula were exact for both the circle and the degenerate flat ellipse.”

The Gauss-Kummer series expressed as a function of the homogenous variable $h = 1 - 4 a b / (a + b)^2$, reads

$$\text{In}[27] = \text{GaussKummer}[h_] = \frac{P_2(a, b)}{a + b}, (a + b) \rightarrow 2 \sqrt{a b} / \sqrt{1 - h}$$

$$\text{Out}[27] = \pi \frac{F_1}{\left(-\frac{1}{2}, -\frac{1}{2}; 1; h\right)}$$
### Series expansions

The series expansion about \( h = 0 \) is useful for small \( h \).

\[
\text{In}[28]:= \quad \text{GaussKummer}[h] + O[h]^3
\]

\[
\text{Out}[28]= \quad \pi + \frac{\pi h}{4} + \frac{\pi h^2}{64} + \frac{25 \pi h^3}{16384} + \frac{49 \pi h^5}{65536} + \frac{441 \pi h^6}{1048576} + \frac{1089 \pi h^7}{4194304} + \frac{184041 \pi h^8}{1073741824} + O(h^9)
\]

Around \( h = 1 \), terms in \( \log(1 - h) \) arise.

\[
\text{In}[29]:= \quad \text{Simplify}[\text{Series}[\text{GaussKummer}[h], \{h, 1, 2\}], 0 < h < 1]
\]

\[
\text{Out}[29]= \quad 4 + (h - 1) + \frac{1}{16} \left( -2 \log(1 - h) - 4 \psi^{(0)}\left(\frac{3}{2}\right) - 4 \gamma + 3 \right) (h - 1)^2 + O((h - 1)^3)
\]

Using functions.wolfram.com/07.23.06.0015.01 we can obtain the general term of this series (c.f. 17.3.33-17.3.36 of [4]).

\[
\text{In}[30]:= \quad \text{GaussKummer}[h] /_2 \text{F}_1(a\_; b\_; c\_; z\_) \rightarrow \text{With}[\{n = c - a - b\},
\]

\[
\frac{\Gamma(a + b + n)}{\Gamma(a) \Gamma(b)} \sum_{k=0}^{n} \frac{(a + n)_k (b + n)_k}{k! (k + n)!} (-\log(1 - z) + \psi(k + 1) + \\
\psi(k + n + 1) - \psi(a + k + n) - \psi(b + k + n)) (1 - z)^k (z - 1)^n + \\
\frac{(n - 1)! \Gamma(a + b + n)}{\Gamma(a + n) \Gamma(b + n)} \sum_{k=0}^{n-1} \frac{(a)_k (b)_k (1 - z)_k}{k! (1 - n)_k} ] / \text{Simplify}
\]

\[
\text{Out}[30]= \quad \frac{1}{4} \left( \sum_{k=0}^{n} \frac{(1 - h)^k \left(\frac{1}{2}\right)_k^2 (-\log(1 - h) + \psi^{(0)}(k + 1) - 2 \psi^{(0)}(k + \frac{1}{2}) + \psi^{(0)}(k + 3))}{k! (k + 2)!} \right) (h - 1)^2 + \\
4 (h + 3)
\]

### Polynomial Approximants

#### Linear Approximant

From the exact values at \( h = 0 \),

\[
\text{In}[31]:= \quad \text{GaussKummer}[0]
\]

\[
\text{Out}[31]= \quad \pi
\]

and at \( h = 1 \),

\[
\text{In}[32]:= \quad \text{GaussKummer}[1]
\]

\[
\text{Out}[32]= \quad 4
\]

one constructs the linear extreme perfect approximant.

\[
\text{In}[33]:= \quad \text{Linear}[h\_] = (1 - h) \text{GaussKummer}[0] + h \text{GaussKummer}[1] // \text{Simplify}
\]

\[
\text{Out}[33]= \quad \pi - h (-4 + \pi)
\]
\textbf{Quadratic Approximant}

The quadratic approximant, exact at \( h = 0, 1/2, 1, \)

\begin{align*}
\text{In[35]:=} & \quad \text{Table}[h, \text{GaussKummer}[h]], \{h, 0, 1, \frac{1}{2}\}] // \text{FullSimplify} \\
\text{Out[35]=} & \quad \left( \begin{array}{c}
\frac{\pi}{2} \\
\frac{\sqrt{\frac{\pi}{2}}}{\Gamma\left(\frac{1}{2}\right)} + \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}}
\end{array} \right)
\end{align*}

\begin{align*}
\text{In[36]:=} & \quad \text{Quadratic}\left[h_\_\right] = \text{InterpolatingPolynomial}\left[\% , \ h\right] // \text{N} \\
\text{Out[36]=} & \quad (0.08918191962 (h - 0.5) + 0.8138163866) h + 3.141592654
\end{align*}

has a maximum absolute relative error of \( \approx 8 \times 10^{-4}. \)

\begin{align*}
\text{In[37]:=} & \quad \text{Plot}\left[10^4 \left(1 - \frac{\text{Quadratic}[h]}{\text{GaussKummer}[h]}\right), \{h, 0, 1\}\right] \\
\text{Out[37]=} & \quad \text{-Graphics-}
\end{align*}

\textbf{\( n \)-order polynomial Approximant}

Here is the \( n \)-order “even-tempered” polynomial approximant, exact at \( h = m/n \) for \( m = 0, 1, \ldots, n. \)

\begin{align*}
\text{In[38]:=} & \quad \text{poly}\left[n_\_\right] := \text{poly}[n] = \text{Function}\left[h, \text{Evaluate}\left[\text{InterpolatingPolynomial}[N@\text{Table}\left\{h, \text{GaussKummer}[h]\right\}, \{h, 0, 1, \frac{1}{n}\}], h\right]\right] \\
\text{The 9th-order approximant has a maximum absolute relative error of} \quad & \quad < 10 \times 10^{-6}.
\end{align*}
\textbf{Chebyshev polynomial Approximant}

Sampling the Gauss-Kummer function at the zeros of $T_n(2x - 1)$, which are at $x_m = \cos^2((m + 1/4) \frac{\pi}{n})$, yields a Chebyshev polynomial approximant.

\texttt{In[40]:= Chebyshevpoly[n_] := Chebyshevpoly[n] = Function[h, Evaluate@Table[cos^2((m + 1/4) \frac{\pi}{n}), GaussKummer[cos^2((m + 1/4) \frac{\pi}{n})], {m, n}]], h]}

The $8^\text{th}$-order approximant has a maximum absolute relative error of $\leq 7 \times 10^{-6}$.

\texttt{Out[41]= -Graphics-}

\textbf{Rational Approximation}

After loading the package (stub),

\texttt{In[42]:= \texttt{\textasciitilde\textasciitilde NumericalMath`}}

one obtains a family of $[N, M]$ rational polynomial minimax approximations.

\texttt{In[43]:= GkApprox[n_, m_] := GkApprox[n, m] = Function[h, Evaluate[MiniMaxApproximation[GaussKummer[h], {h, {0, 1}, n, m}][2, 1]]]}

For example, the [4,3] minimax approximation,
This leading optimal one
After uniformly approximant leads Using
has (absolute) relative error \( \leq 2.3 \times 10^{-7} \), but is not “extreme perfect”.

\[
\text{GKapprox[4, 3][} h] = \frac{-0.08111828562 h^4 + 0.273498199 h^3 + 1.771628564 h^2 - 5.055401264 h + 3.14159195}{-0.1414596605 h^3 + 1.013205136 h^2 - 1.859195682 h + 1}
\]

has (absolute) relative error \( \leq 2.3 \times 10^{-7} \), but is not “extreme perfect”.

\[
\text{In[45]:=} \quad \text{Out[44]= } \quad \text{Plot} \left[ 10^7 \left( 1 - \frac{\text{GKapprox[4, 3][} h]}{\text{GaussKummer[} h]} \right) \right], \{h, 0, 1\}
\]

\[
\text{Out[45]=} \quad \text{Graphics} \quad \text{Using the linear approximant, } 4 h + \pi (1 - h), \text{and noting that } h (1 - h) \text{ vanishes at both } h = 0 \text{ and } h = 1, \text{leads to an optimal } [N + 2, M] \text{ extreme perfect approximant of the form}
\]

\[
\pi_2 F_1 \left( -\frac{1}{2}, -\frac{1}{2}; 1; h \right) \approx 4 h + \pi (1 - h) + \alpha (1 - h) \prod_{p=1}^N (h - p_j) \prod_{q=1}^M (h - q_j),
\]

where the parameters \( \alpha, \{p_i\}_{i=1, \ldots, N}, \text{ and } \{q_j\}_{j=1, \ldots, M} \text{ need to be determined. Implementation of the approximant is immediate.}

\[
\text{In[46]:=} \quad \text{EllipseApproximant}[\alpha, \{p_i\}, \{q_j\}] := 
\]

\[
\text{Function}[h, \text{Evaluate}[4 h + \pi (1 - h) + \alpha (1 - h) \frac{\text{Times} @ (h - p)}{\text{Times} @ (h - q)}]]
\]

After uniformly sampling the Gauss-Kummer function,

\[
\text{In[47]:=} \quad \{xdata, ydata\} = \text{Table}[[h, \text{GaussKummer[} h]], \{h, 0, 1, 0.001\}] // \text{Transpose};
\]

one can use NMinimize and the \( \infty \)-norm to obtain the accurate approximants. For example, the (almost) optimal \([3, 2]\) approximant is computed using

\[
\text{In[48]:=} \quad \text{NMinimize}[[ydata - \text{EllipseApproximant}[\alpha, \{p\}, \{q_j\}][xdata]], \{\alpha, 0.22, 0.24\}, \{p, 1.25, 1.35\}, \{q, 3.4, 3.5\}, \{r, 1.15, 1.25\}]
\]

\[
\text{Out[48]=} \quad \{0.0000140975141, \{p \rightarrow 1.285457885, q \rightarrow 3.475000451, r \rightarrow 1.196711294, \alpha \rightarrow 0.2354557322\}\}
\]

leading to

\[
\text{In[49]:=} \quad \text{EllipseApproximant}[\alpha, \{p\}, \{q_j\}][h] \quad \text{/ Last[\%]}
\]

\[
\text{Out[49]=} \quad \frac{0.2354557322 (h - 1.285457885) h (1 - h)}{(h - 3.475000451) (h - 1.196711294) + \pi (1 - h) + 4 h}
\]

This simple approximant has (absolute) relative error \( \leq 4 \times 10^{-8} \).
Conclusions

*Mathematica* is an ideal tool for developing accurate approximants to special functions because:

- all special functions of mathematical physics are built-in and can be evaluated to arbitrary precision for general complex parameters and variables;
- standard analytical methods—such as symbolic integration, summation, series and asymptotic expansions, and polynomial interpolation—are available;
- properties of special functions—such as identities and transformations—are available at MathWorld [6] and the Wolfram functions Site [7] and, because these properties are expressed in *Mathematica* syntax, can be used directly;
- relevant built-in numerical methods include rational polynomial approximants, minimax methods, and numerical optimization for arbitrary norms;
- visualization of approximants can be used to estimate the quality of approximants; and
- combining these approaches is straightforward and leads, in a natural way, to optimal approximants.

This paper uses the exercise of computing the perimeter of an ellipse using a simple set of approximants to illustrate these points.

References


