Fractals:

an Introduction through Symmetry

for beginners to fractals,
highlights magnification symmetry
and fractals/chaos connections

I hope viewers enjoy this gentle approach to math education. The focus is Math-Art. It is sensory, heuristic, with the intent of imparting *perspective* as opposed to strict knowledge per se. Ideally, viewers new to fractals will walk away with an ability to recognize some fractals in everyday settings accompanied by a sense of how fractals affect practical aspects of our lives, looking for connections between math and nature.

This personal project was put together with the input of experts from the fields of both fractals and chaos:

Academic friends who provided input:

Don Jones
Department of Mathematics & Statistics
Arizona State University

Reimund Albers
Center for Complex Systems & Visualization (CeVis)
University of Bremen

Paul Bourke
Centre for Astrophysics & Supercomputing
Swinburne University of Technology

A fourth friend who has offered feedback, whose path I followed in putting this together, and whose influence has been tremendous prefers not to be named yet must be acknowledged. You know who you are. Thanks. ☺

First discussed will be three common types of symmetry:

- Reflectional (Line or Mirror)
- Rotational (N-fold)
- Translational

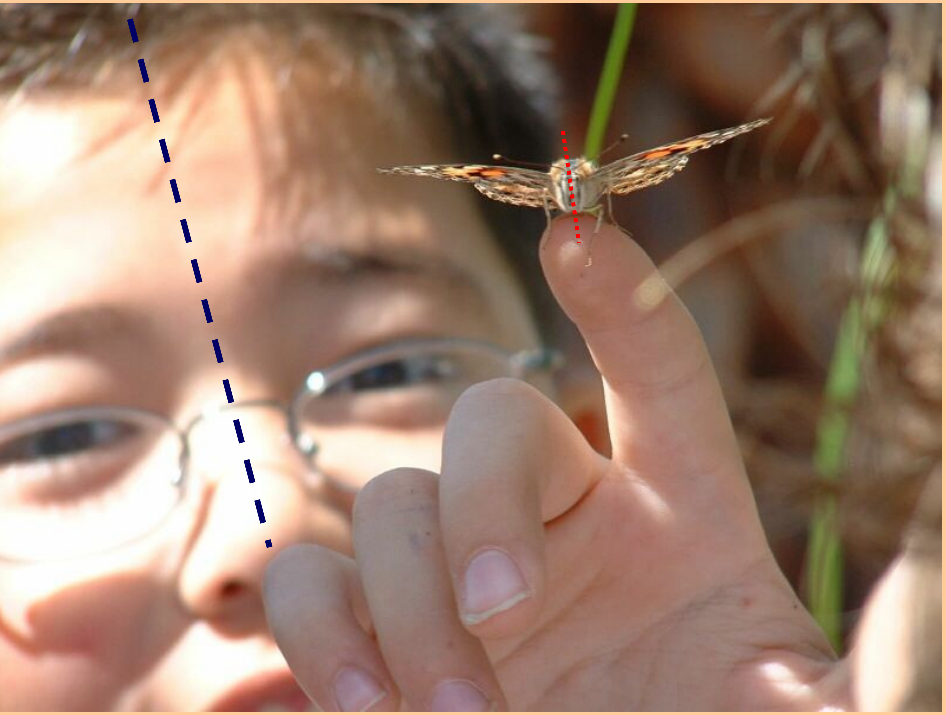
and then: the Magnification (Dilatational a.k.a. Dilational) symmetry of fractals.

Reflectional (aka Line or Mirror) Symmetry

A shape exhibits reflectional symmetry if the shape can be bisected by a line L, one half of the shape removed, and the missing piece replaced by a reflection of the remaining piece across L, then the resulting combination is (approximately) the same as the original.[1]

In simpler words, if you can fold it over and it matches up, it has reflectional symmetry. This leaf, for instance, and the butterfly caterpillar sitting on it, are roughly symmetric. So are human faces. Line symmetry and mirror symmetry mean the same thing.

From "An Intuitive Notion of Line Symmetry"



The butterfly and the children have lines of reflection symmetry where one side mirrors the other.

Taken at the same time at the Desert Botanical Gardens Butterfly Pavilion, the little butterfly is a Painted Lady *(Vanessa, cardui).* Its host plant is *(Thistles, cirsium).*

These images have lines of symmetry at the edge of the water.

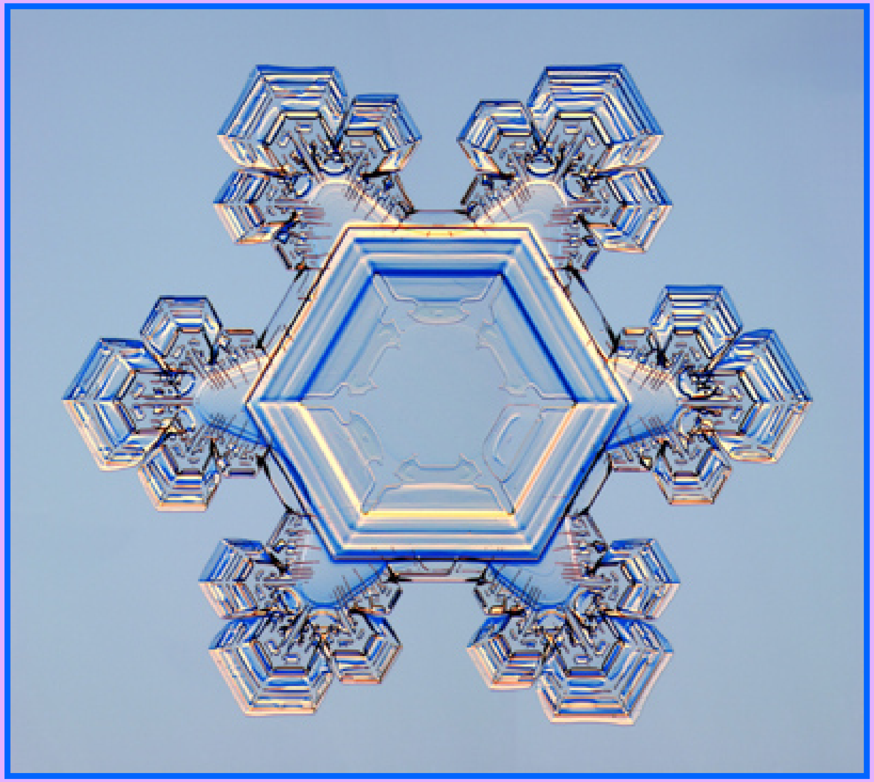
Here is a link to a [PowerPoint presentation]“created by Mrs. Gamache using the collection of web pages by the Adrian Bruce and students of 6B.”

This site lets you [create your own symmetry patterns!] Choose your type and color, then start moving the mouse and clicking.

Rotational (N-fold) Symmetry

A shape exhibits rotational symmetry if rotation about some center point returns the shape to its original configuration.[2]

Libbrecht of Caltech discusses symmetry of ice crystals and snowflakes

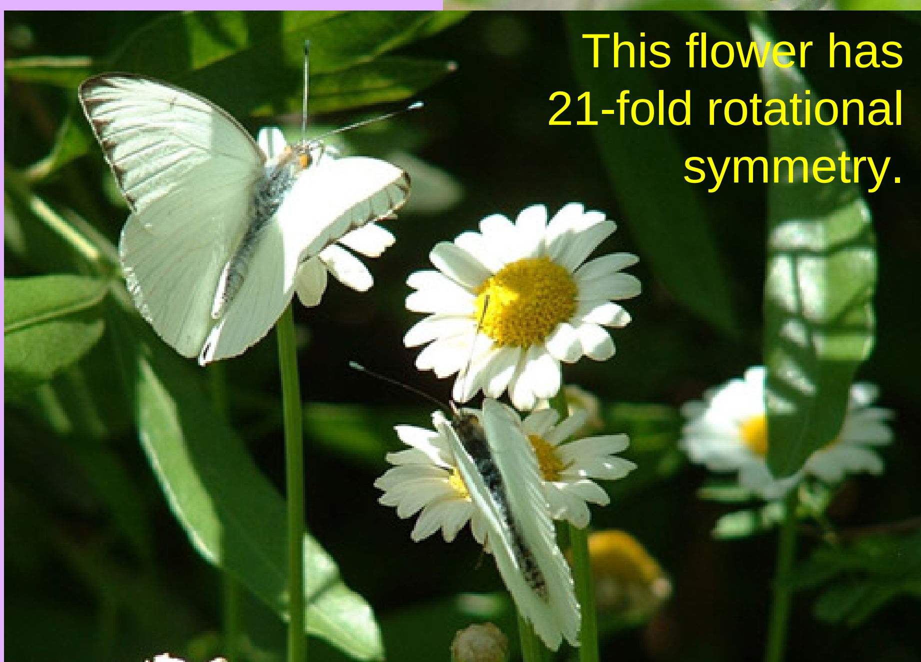


Real snowflake image taken by Kenneth Libbrecht using a special photo microscope


These blooms have 5-fold rotational symmetry. They can be turned 5 times to leave the figure unchanged before starting over again.
A pentagon also has 5-fold symmetry.
The butterfly is a Spicebush Swallowtail (*Papillo, troilus*).


An example of 4-fold rotational symmetry,
a property shared by the square.


5-fold or 6-fold
symmetry here


This flower has
21-fold rotational
symmetry.


The tiny blooms have 4-fold
symmetry. Question: does
the spherical bloom they sit
on have n-fold symmetry?

Butterfly names

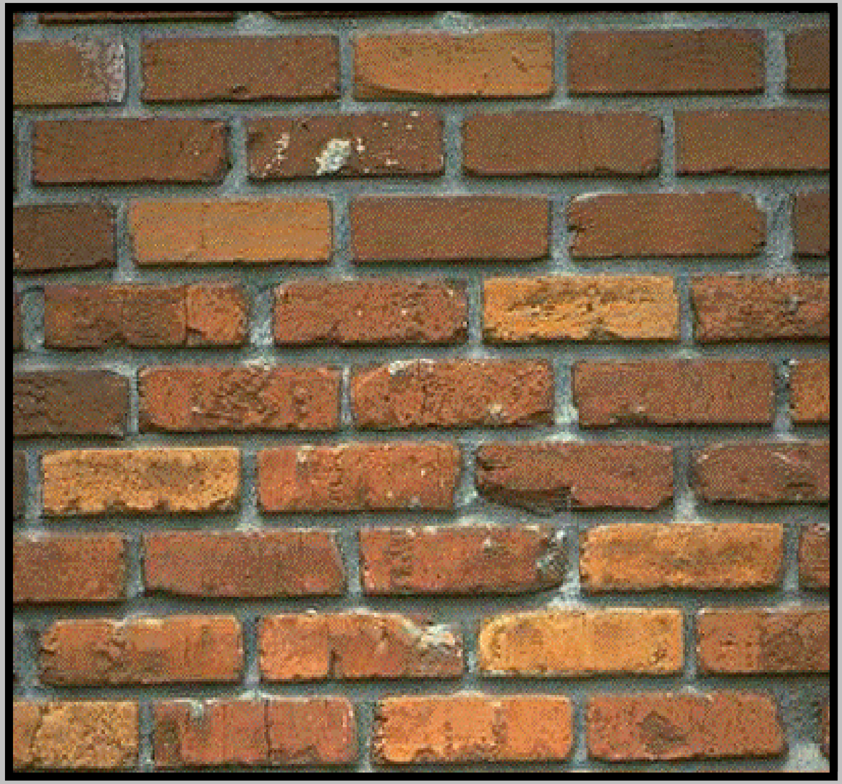
Translational Symmetry

A shape exhibits translational symmetry if displacement in some direction - horizontal or vertical, for example - returns the shape to (approximately) its original configuration.[3]

The bricks in the image have translational symmetry.

Also, the image of the bricks will have translation symmetry when sliding, provided there is no rotation during the move.

Orientation must be preserved while translating.



Magnification (Dilatational) Symmetry

Less familiar is symmetry under magnification: zooming in on an object leaves the shape approximately unaltered.[4]

Zooming in on a fractal object leaves the shape approximately unaltered.

Fractals exhibit magnification symmetry.

Types of Fractals highlighted:

- Natural
- Geometric
- Complex and Random

Brief discussion: Frames of Reference

and examples of:

Exponential Growth

Fractals Across the Disciplines

Natural Fractals

Multifractals

Chaos

Natural fractals have a limited number of stages of growth, and the growth between stages shows variation. They have connections to Multifractals and Chaos theory.

Fractals in the Biological Sciences
by N. C. Kenkel and D. J. Walker
University of Manitoba
Quantitative Plant Ecology Laboratory

below is an excerpt from their web page

Section 5.7 Chaos and Time Series Analysis:

"Chaos, which is closely related to fractal geometry, refers to a kind of constrained randomness (Stone and Ezrati 1996). Wherever a chaotic process has shaped an environment, a fractal structure is left behind."


Photo by Gayla Chandler
Post Processing by Kim Letkeman

“Fractal geometry was designed to handle shapes that appear complicated, but with complexity arranged in some hierarchical fashion. So at a minimum, fractals must have some substructure.”

(Michael Frame, Yale University)

One necessary condition for fractal substructure is the same or a highly similar shape between a minimum of 3 stages of growth (there exists disagreement on this, especially between disciplines).

The entire fern family reveals self- similarity: successive stages of growth that closely resemble earlier stages.

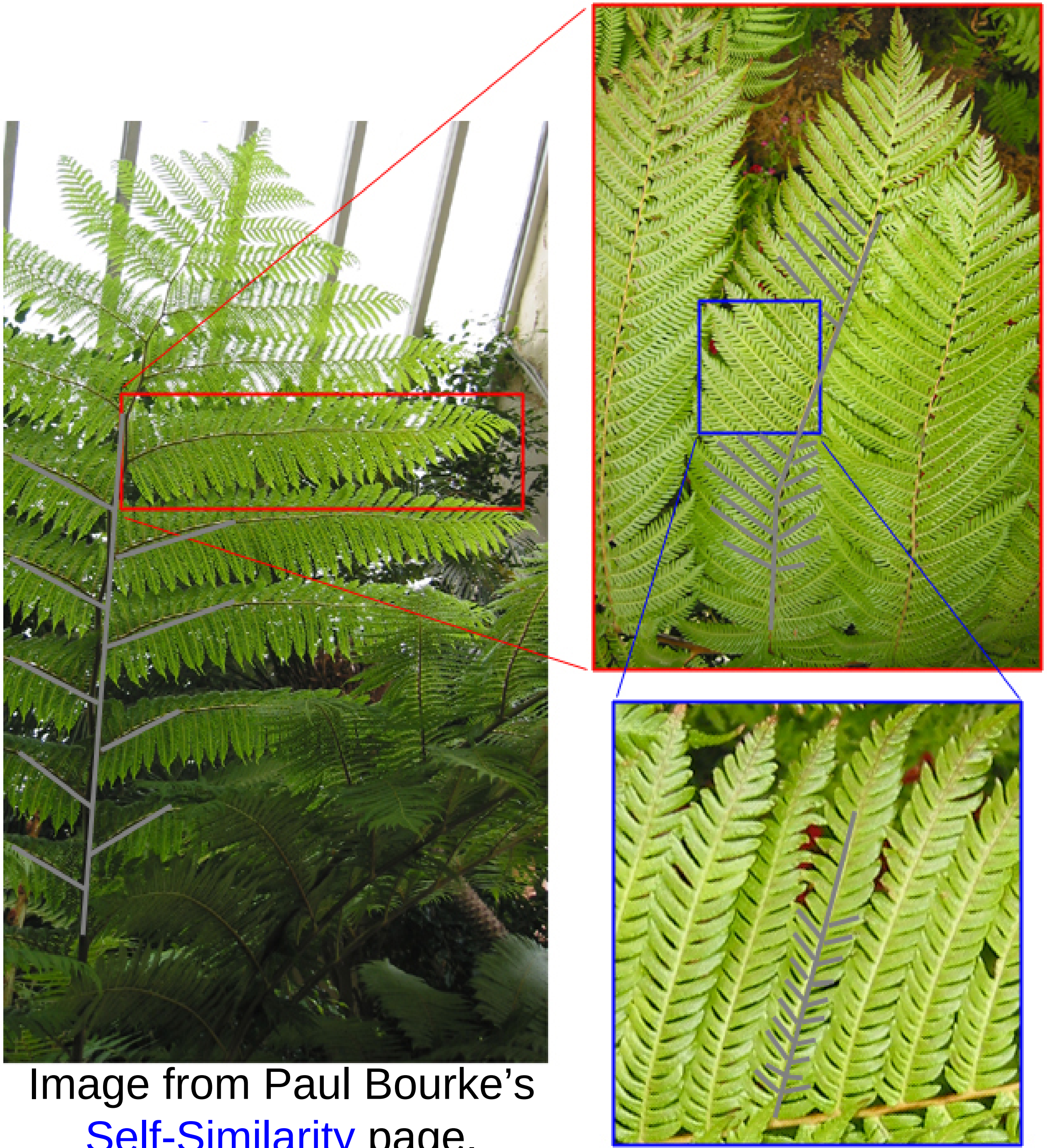


Image from Paul Bourke's Self-Similarity page.

Photo courtesy of Pbase artist Ville Vels of Estonia

Photo courtesy of Pbase artist Ville Vels of Estonia


Repeating shapes on different scales.
Photo courtesy of Pbase artist Ville Vels of Estonia

Photo courtesy of Pbase artist Ville Vels of Estonia

Branching Patterns

This corrosion cast of the liver used in medical research reveals fractal branching as do several body organs. Most branching in nature is fractal: leaf veins and rivers, our circulatory system and lightning, to name a few.

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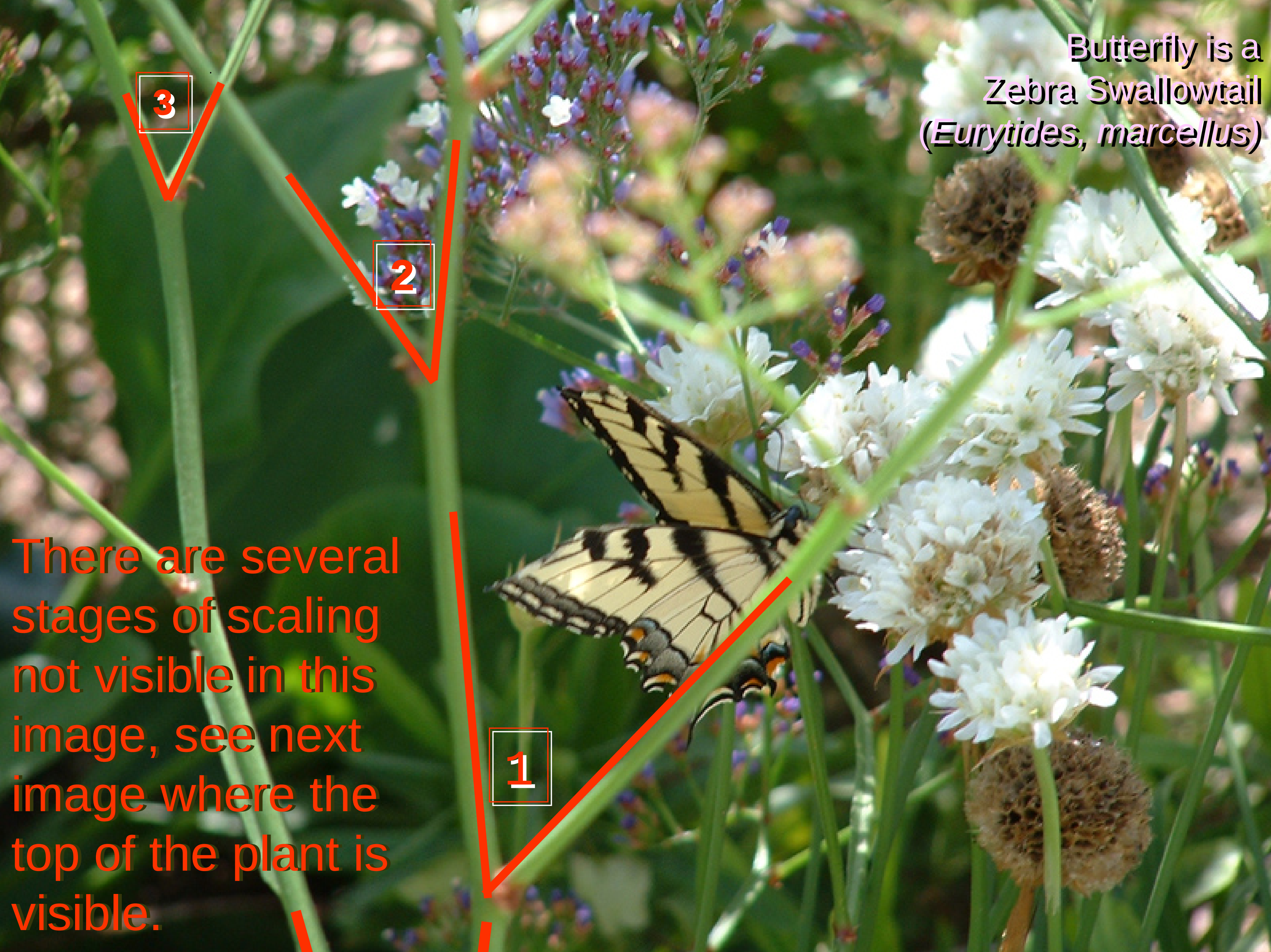
Butterfily is a
Zebra Swallowtail
(*Eurytides, marcellus*)
3
2
1
There are several stages of scaling not visible in this image, see next image where the top of the plant is visible.


The butterfly is a
Great Southern White
The scaled
branching extends
upward throughout
the plant. A small
branch, if magnified,
would look like a
larger branch.


Also notice how entire sections of the plant resemble each other on different scales. The structure of smaller sections dictates the shape of larger sections.


Fractal branching is captured in shadow below. From this view, again notice how the parts resemble the whole.
A Painted Lady is present!

The plant in the previous slides resembles this computer-generated binary fractal tree.

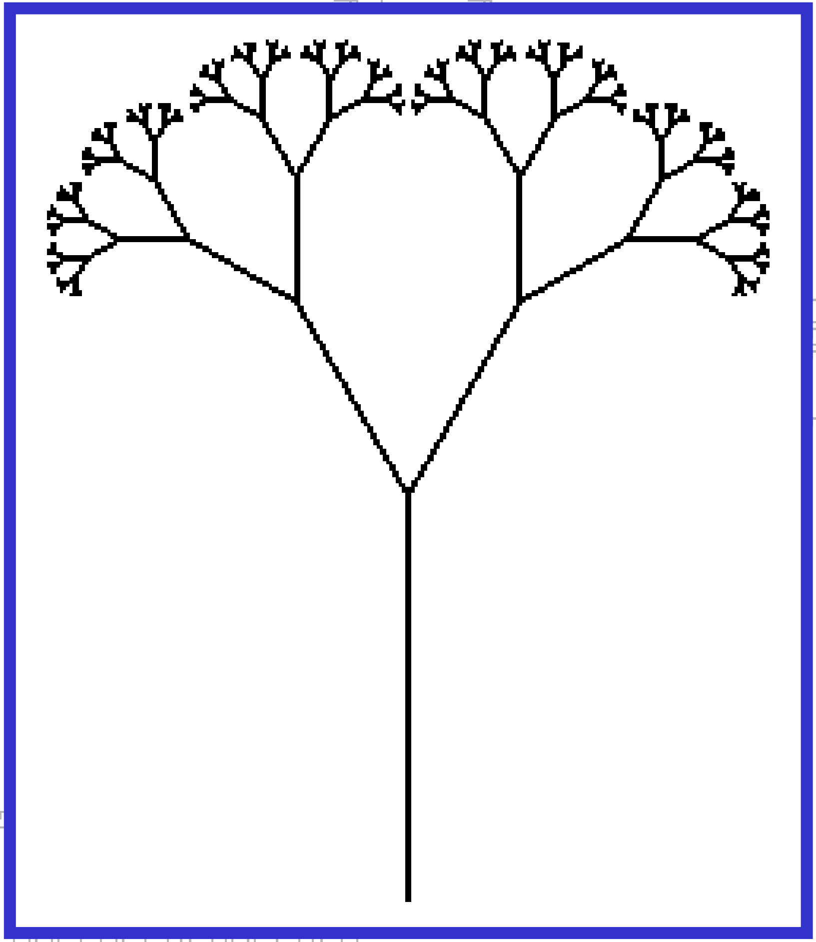

Image by Don West

Photograph courtesy of Clive Pierson, taken in Ireland.


Similarity in tree branches can be seen in the shapes of clumps of tree, intersections of branches, the flow of limbs... Repeating a simple pattern throughout growth yields a complex structure with magnification symmetry, a structure made of parts that are similar to the whole: a *fractal* structure.
Photograph courtesy of Clive Pierson, taken in Ireland.

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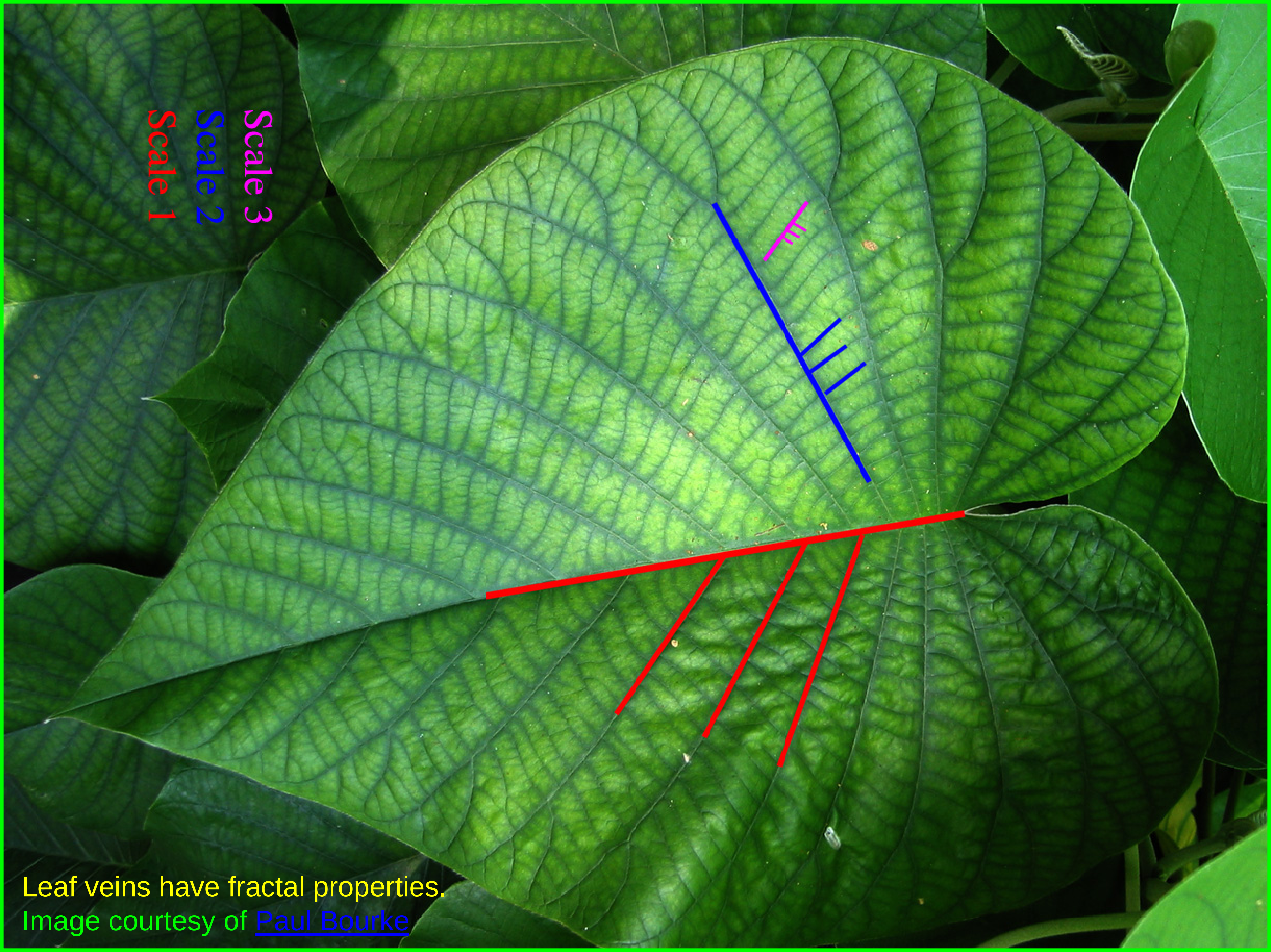

Leaf veins have fractal properties.
Image courtesy of Paul Bourke

With fractals, the structure behind small sections dictates overall shape.

We have seen empirical verification of this in previous examples, how bigger shapes were aggregations of the smaller shapes that made them up. This is also true of clouds, mountains, ocean waves, lightning, and many other aspects of nature. An ocean wave is made up of a lot of little waves, which are in turn made up of yet smaller waves. This is why fractal equations tend to be simple. Tremendous complexity can result from iterating simple patterns.

Of those aspects that have an embedded fractal structure, their fractal aspect only describes properties of shape and complexity. Read this Word of Caution from Nonlinear Geoscience: Fractals. They refer to randomness that is taken into account in Multifractal theory, which has ties to Chaos theory and Nonlinear Dynamics.

The Yale Fractal Geometry website points out Common Mistakes in Finding Fractals. Also view this Introduction to Fractals PowerPoint presentation out of Florida Atlantic University by Liebovitch and Shehadeh that makes many fractal/nonfractal comparisons. Read a paper from Complexity International about language issues with regard to fractals: Is There Meaning In Fractal Analysis?

This is a Sweet Acacia (Acacia, smallii) tree. Its unbloomed flower appears to be a sphere made up of smaller-scale spheres, but take a closer look:

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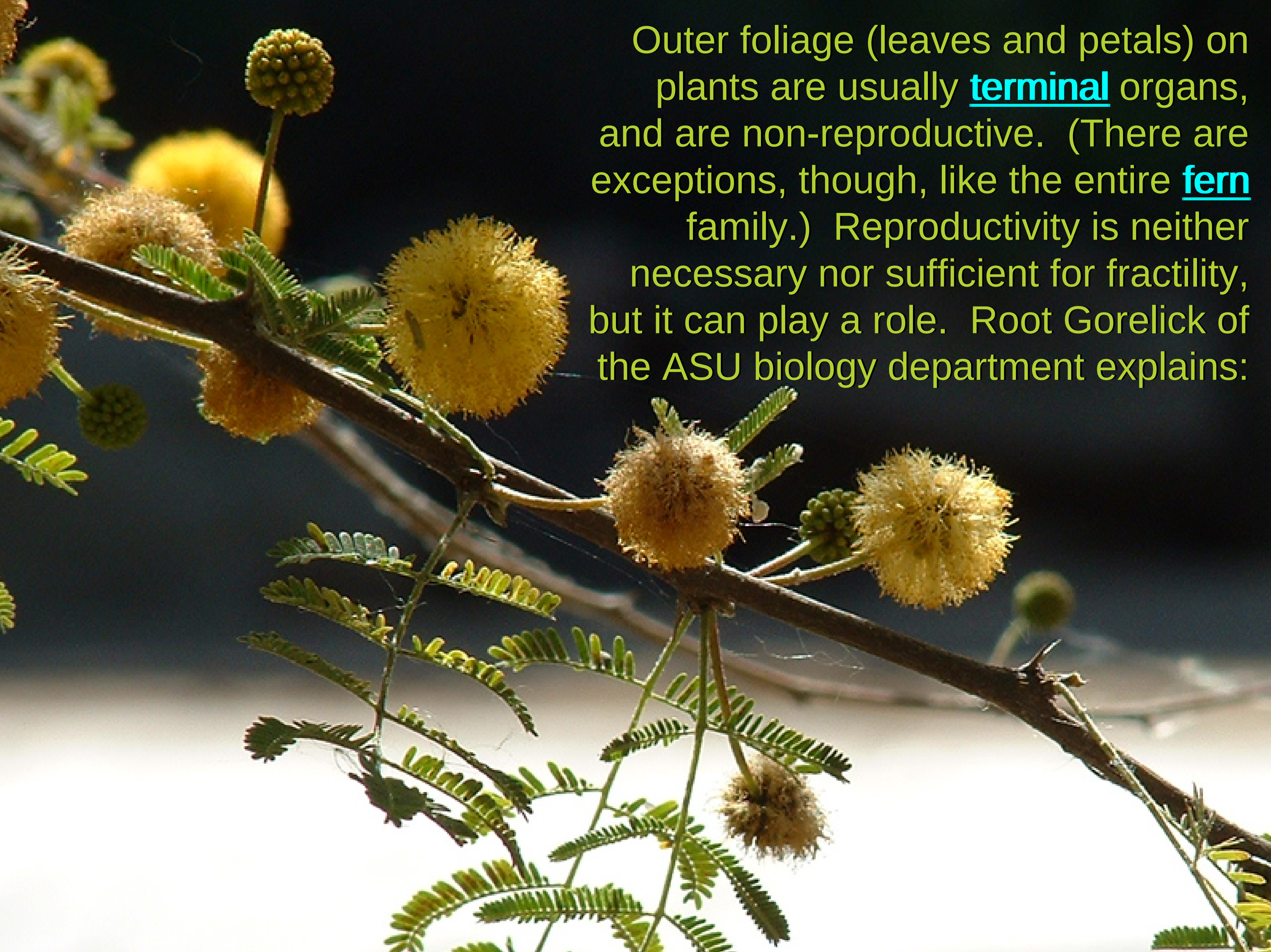

Outer foliage (leaves and petals) on plants are usually **terminal** organs, and are non-reproductive. (There are exceptions, though, like the entire **fern** family.) Reproductivity is neither necessary nor sufficient for fractility, but it can play a role. Root Gorelick of the ASU biology department explains:

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"Leaves are terminal organs, hence don't reproduce miniature copies of themselves as do stems, roots, and many reproductive structures. Therefore, I expect leaves to be least fractal of these organs." (Root Gorelick)

Geometric Fractals

I like to compare Geometric fractals to objects/systems in a vacuum in physics. They are, as their name suggests, geometric constructs, perfect (Ideal) systems exempt from internal deviations or potential changes from outside influences (other than human error in constructing them).

I haven’t included Complex fractals such as the Mandelbrot Set and Julia Sets in the Geometric fractals category. Complex fractals are mentioned later.

The Sierpinski Tetrahedron
Fractal type: Geometric

Tetrahedra are increasing in number in powers of 4
Tetrahedra are decreasing in edge-length in powers of ½
Volume is decreasing in powers of ½

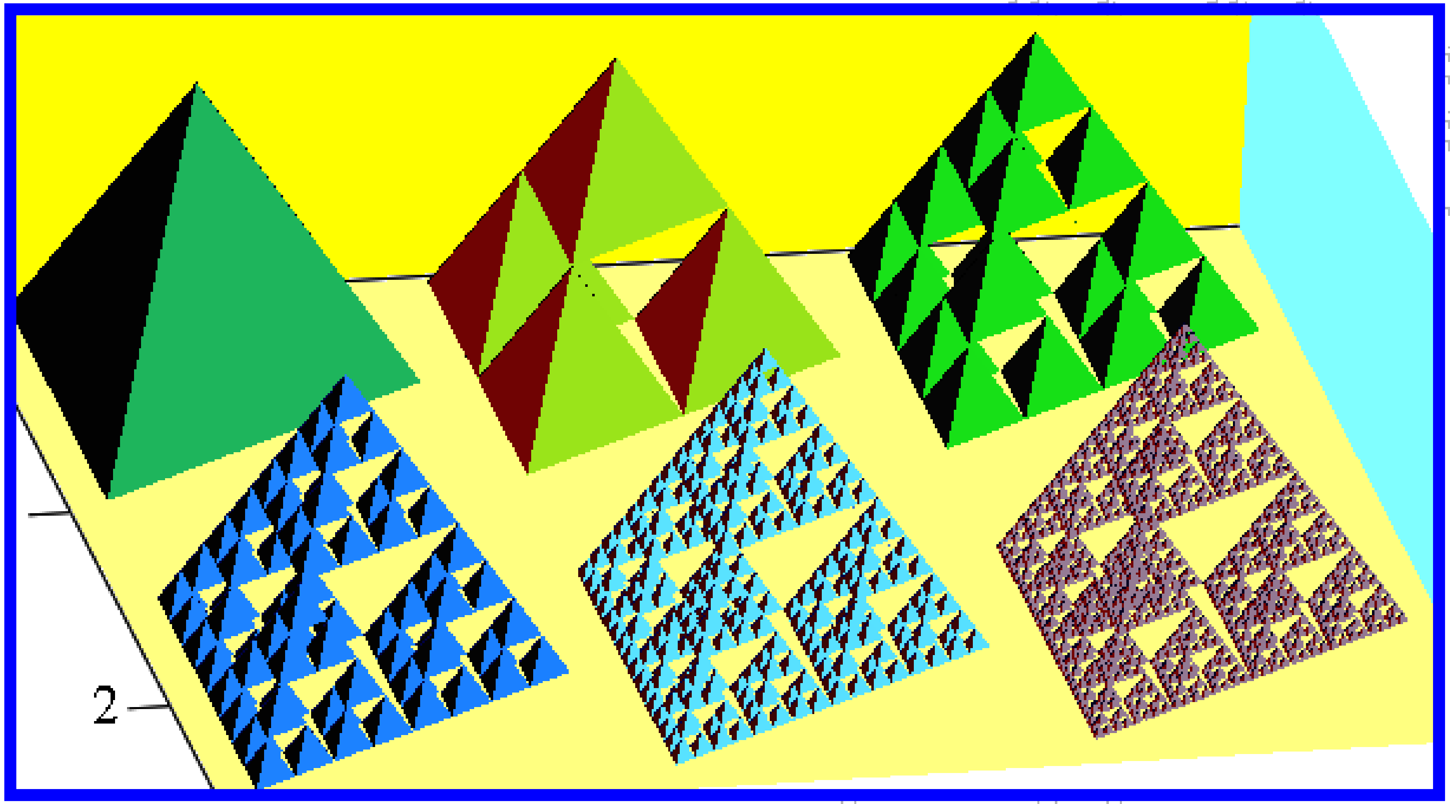
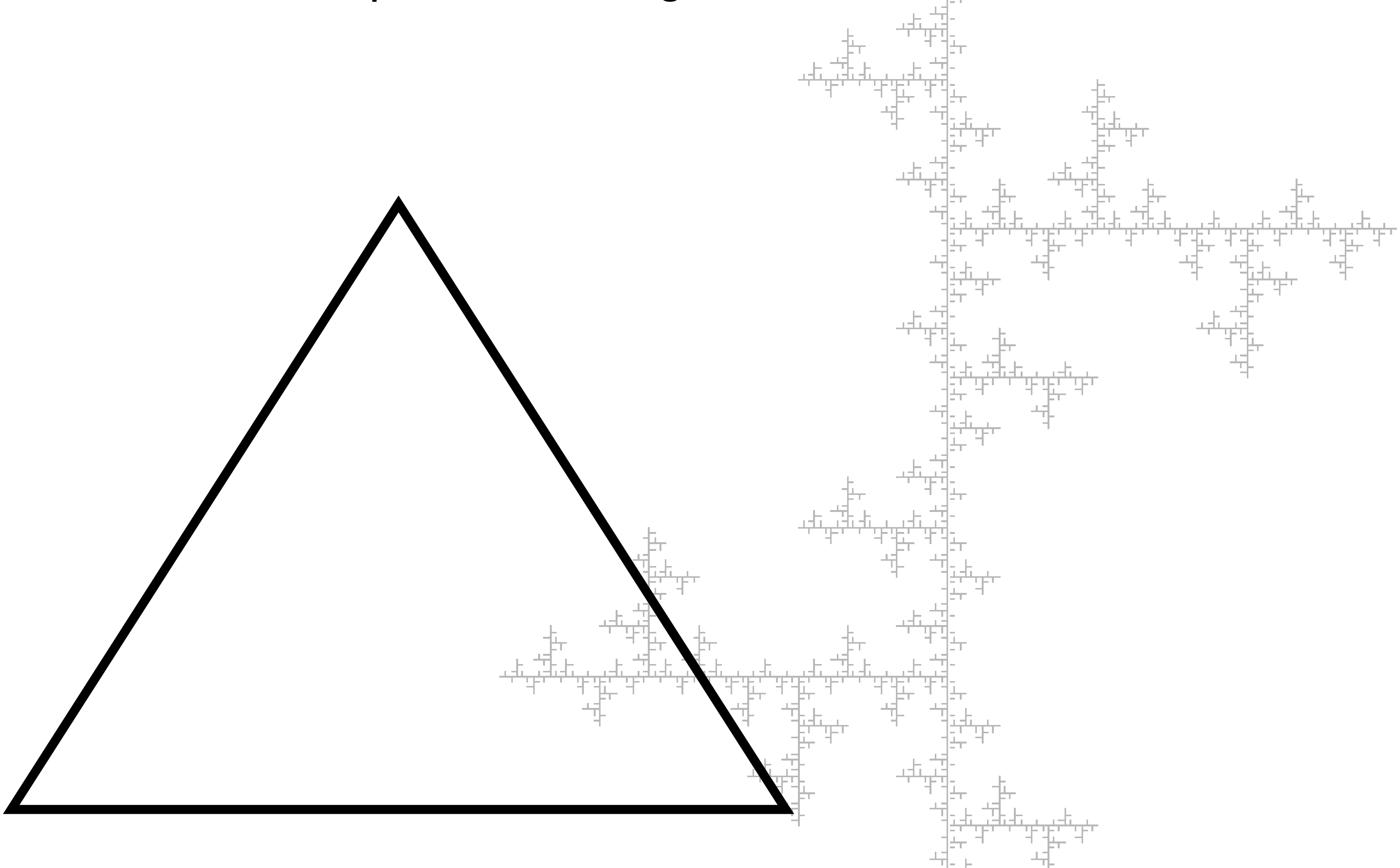


Image created using MathCad by Byrge Birkeland of Agder University College, Kristiansand, Norway

To consider this fractal, it is important to know something about a tetrahedron.

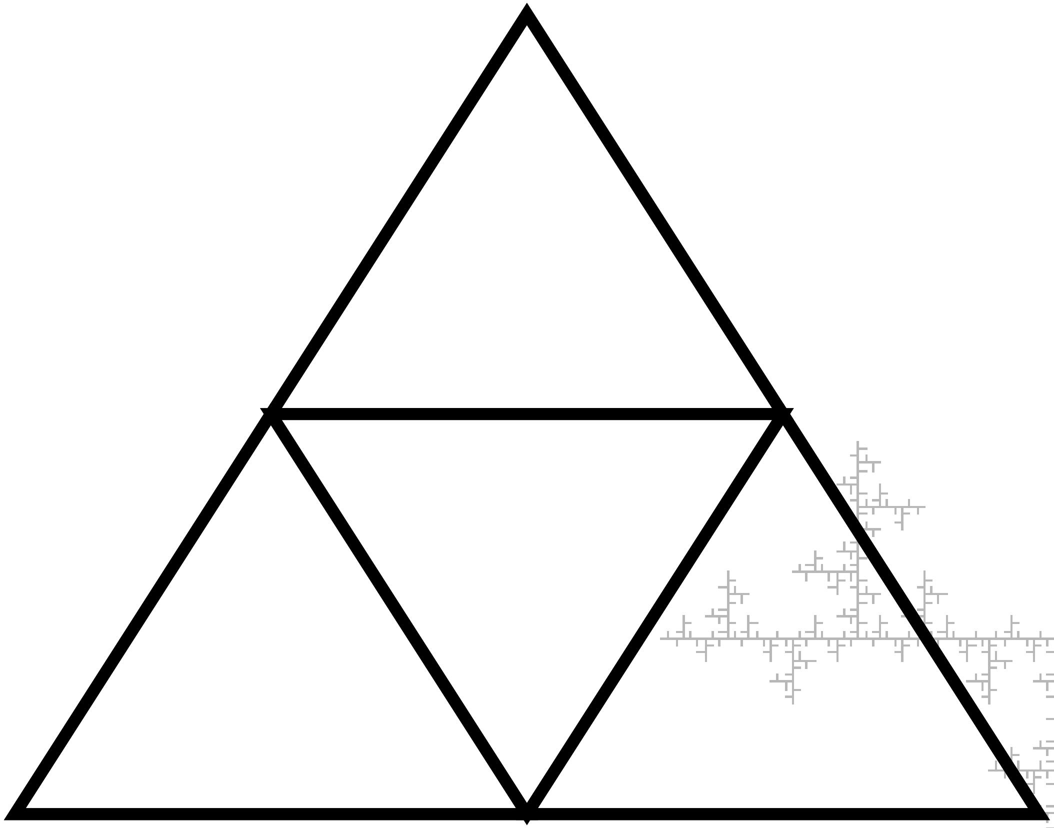
- Start with an equilateral triangle.



To consider this fractal, it is important to know something about a tetrahedron.

- Start with an equilateral triangle.

- Divide it into 4 equilateral triangles by marking the midpoints of all three sides and drawing lines to connect the midpoints.



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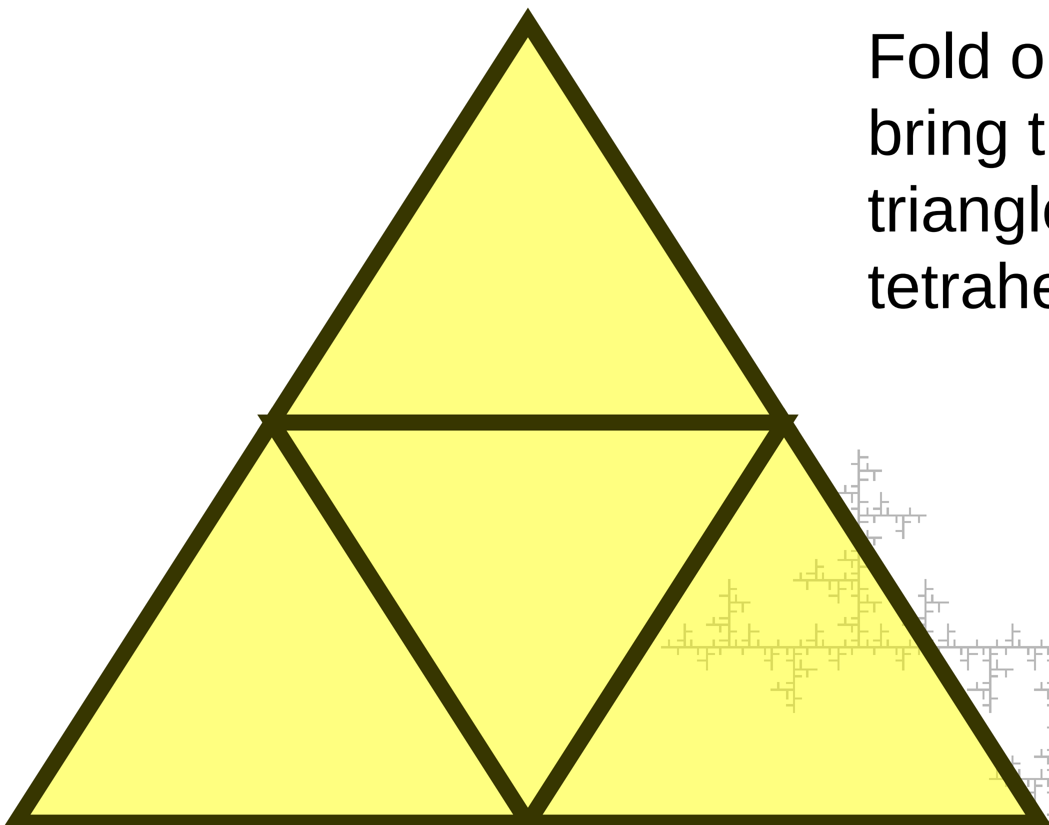
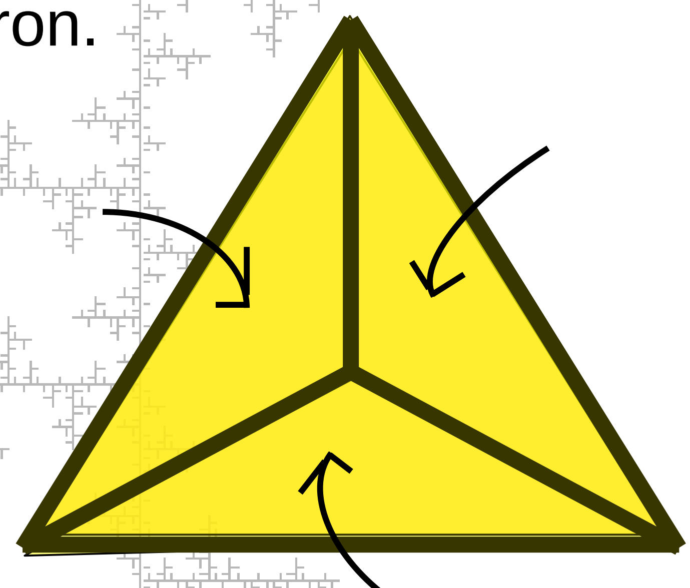
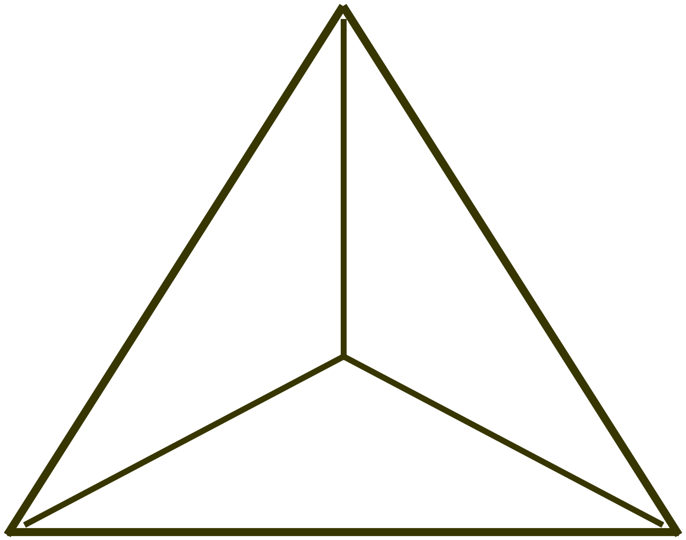


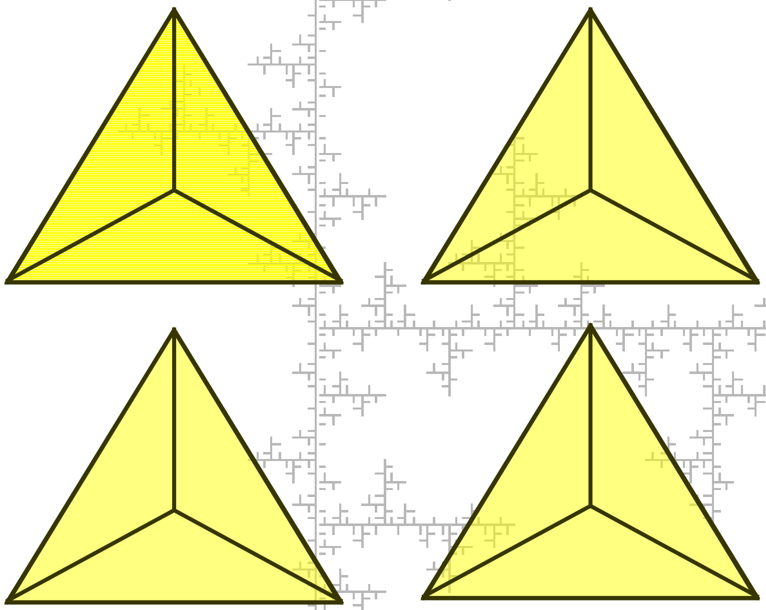
Fold on the midpoint lines and bring the tips of the equilateral triangle together to make a tetrahedron.



To build a stage-1:



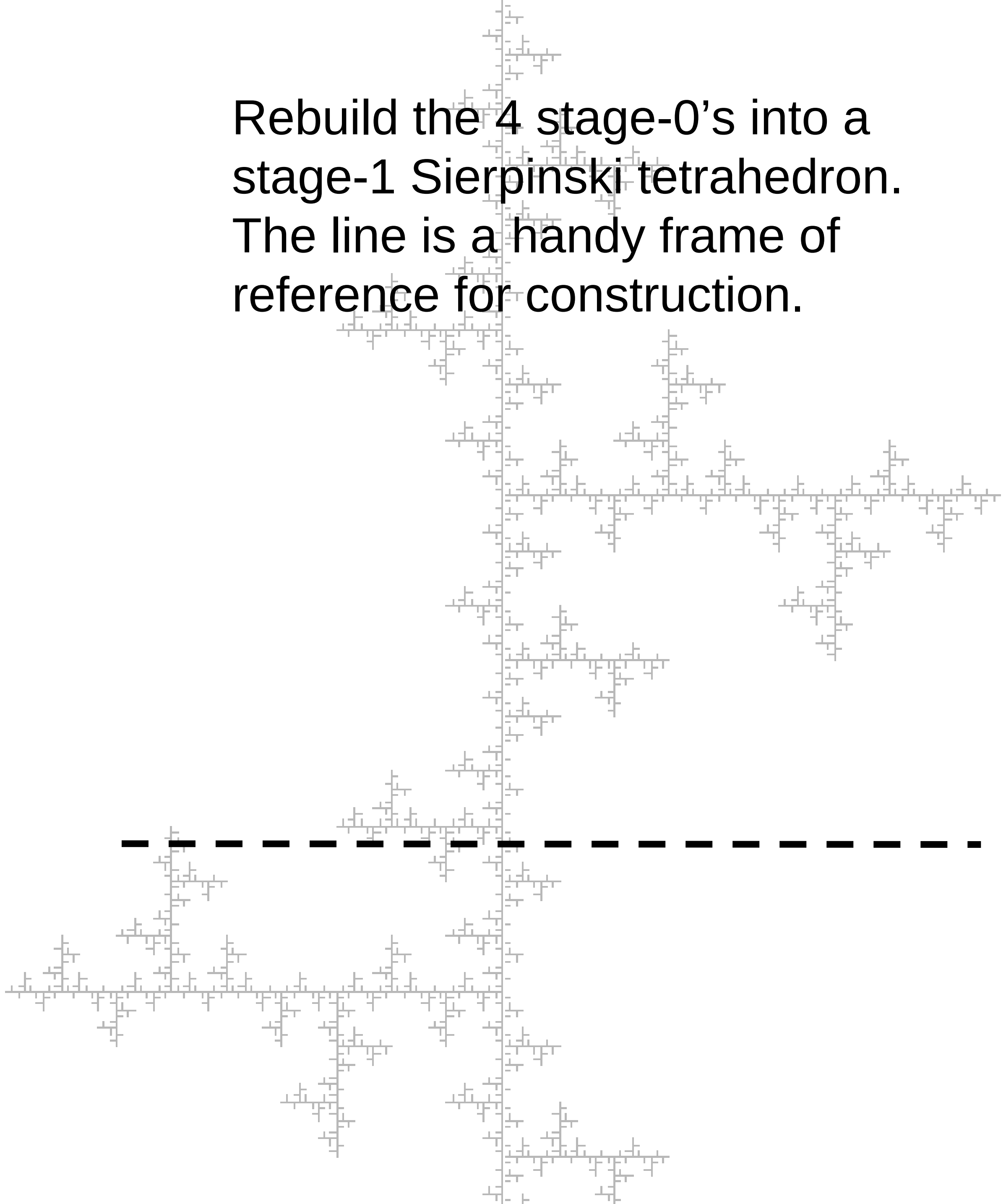
Start with a regular tetrahedron. It is called the stage-0 in the Sierpinski tetrahedron fractal family.



Reduce it by a factor of 1/2

Replicate (4 are needed). The tetrahedra are kept transparent on this slide to reinforce that these are tetrahedra and not triangles.

Rebuild the 4 stage-0's into a stage-1 Sierpinski tetrahedron. The line is a handy frame of reference for construction.



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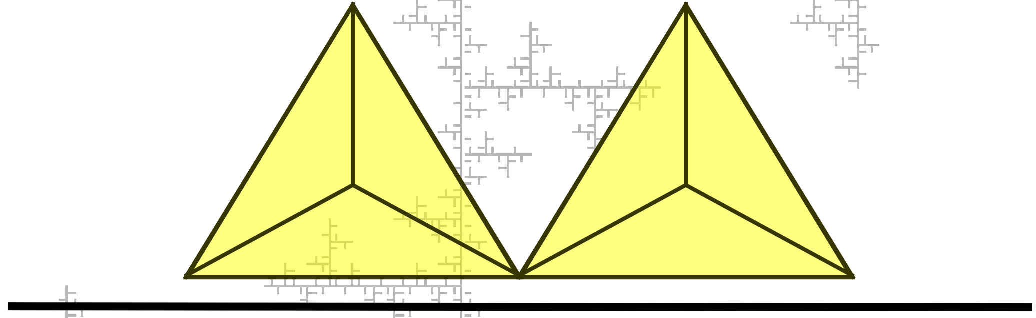

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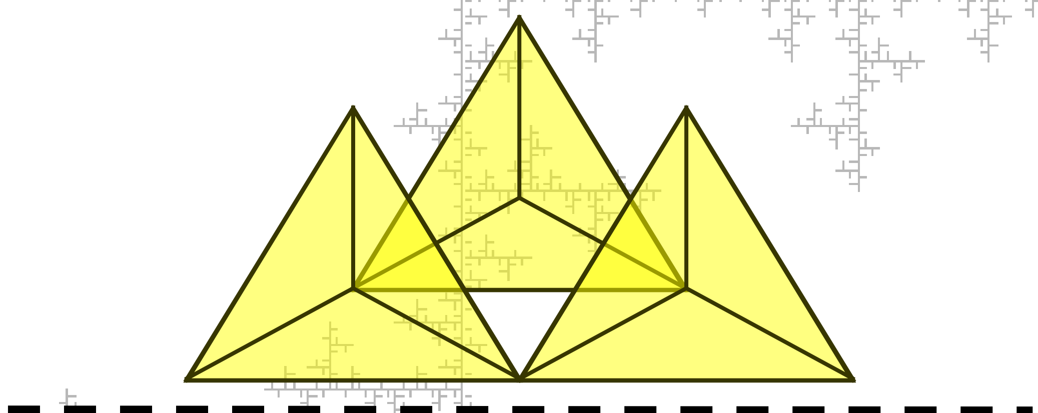

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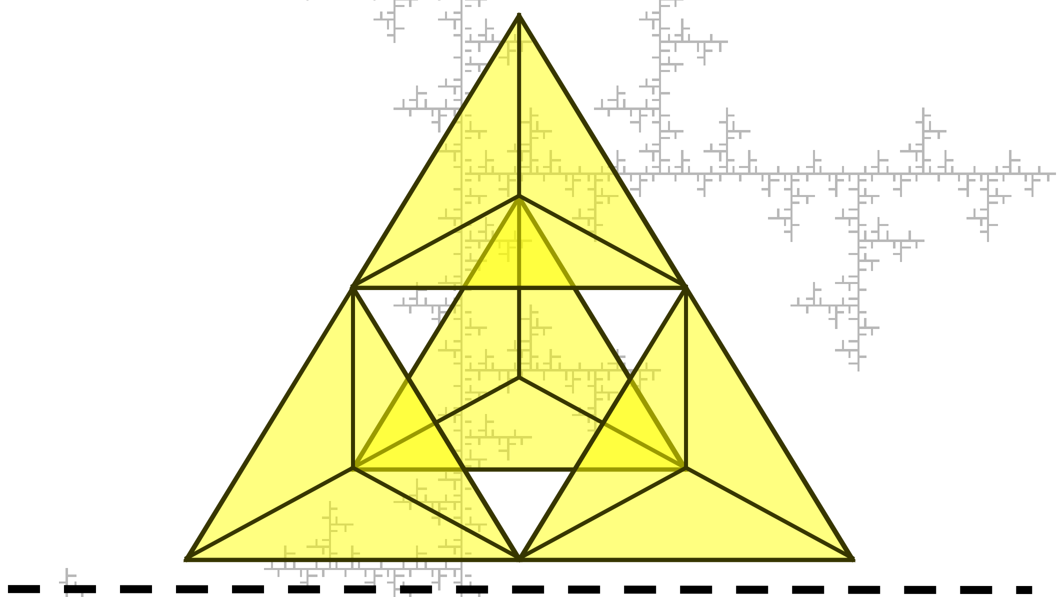
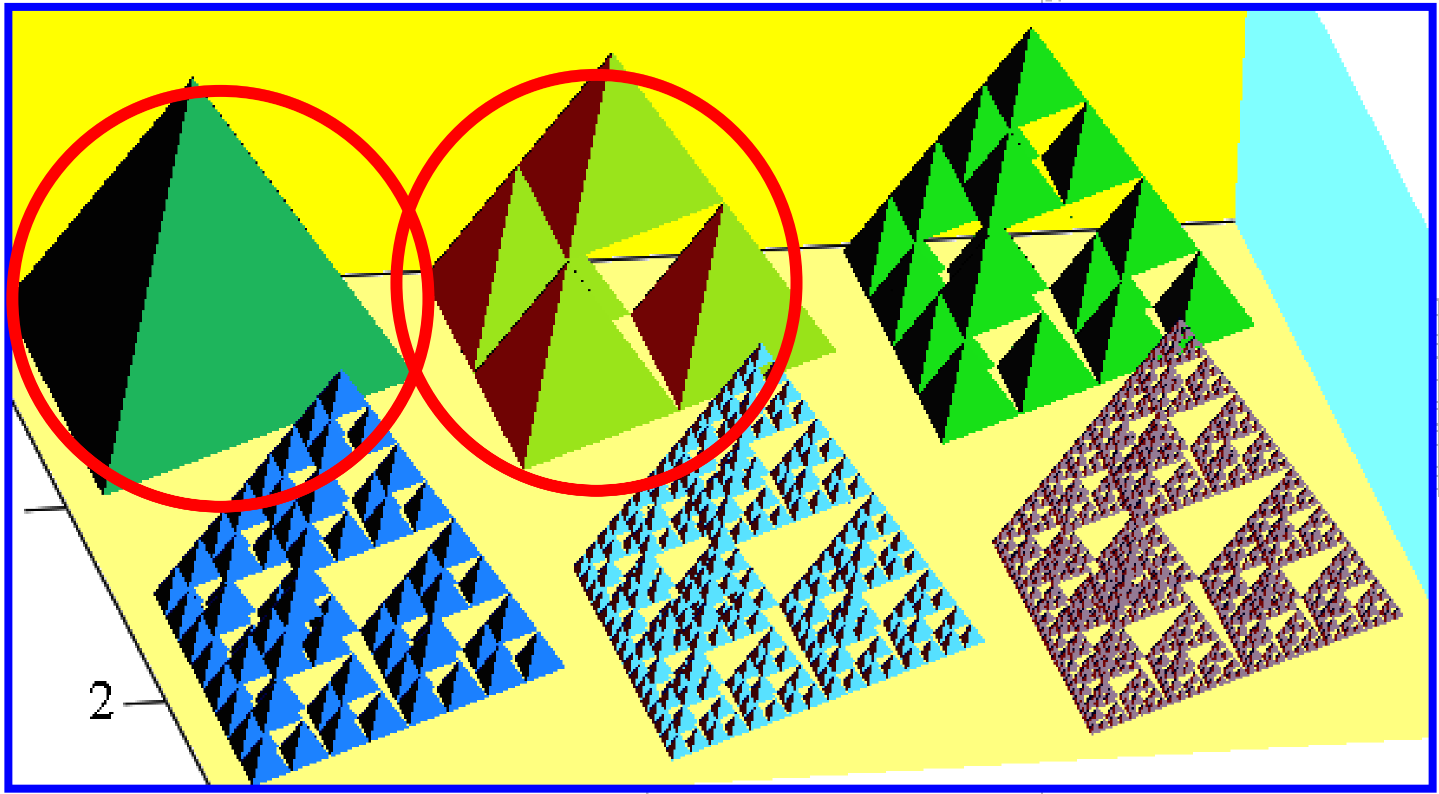
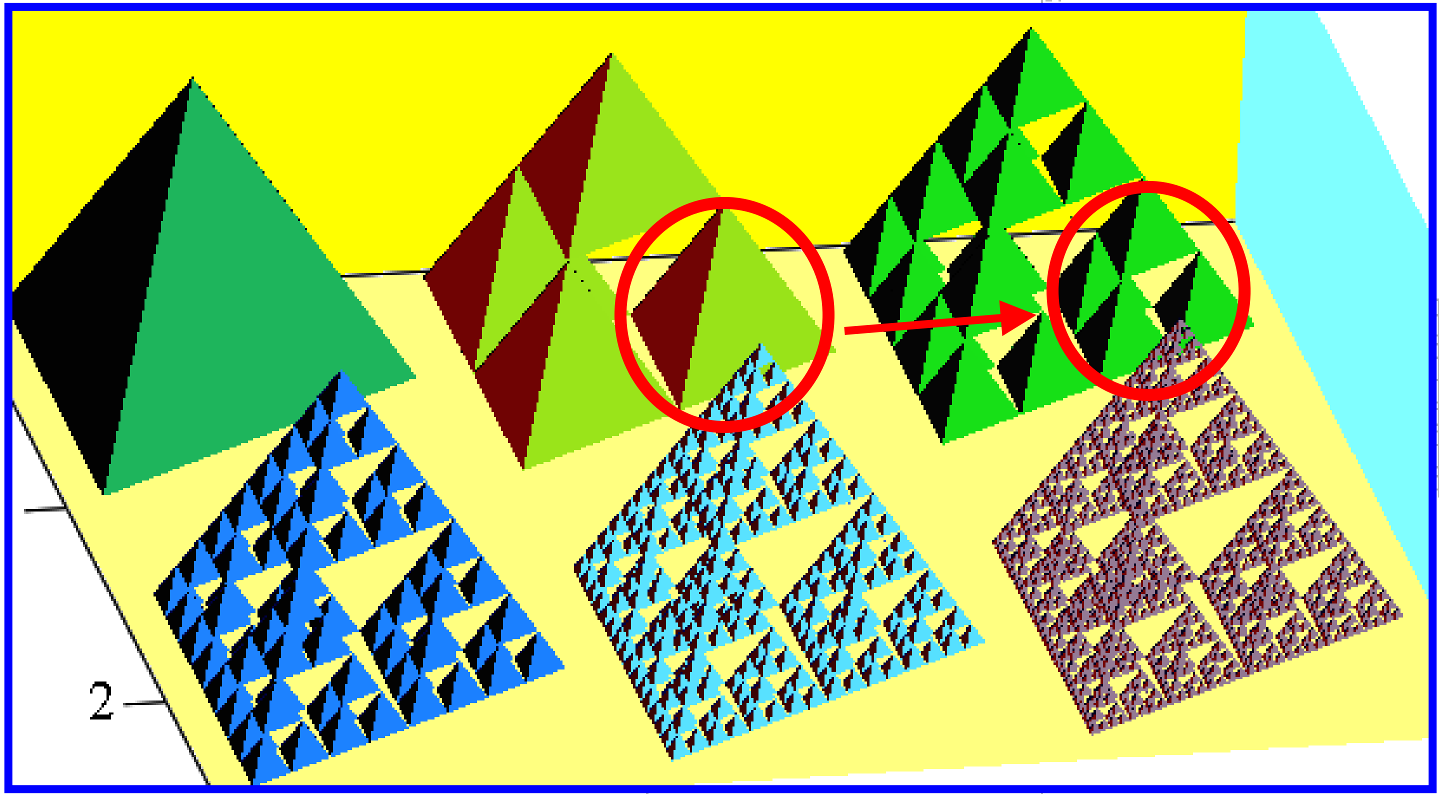


Revisiting the earlier image, notice that each tetrahedron is replaced by 4 tetrahedra in the next stage.

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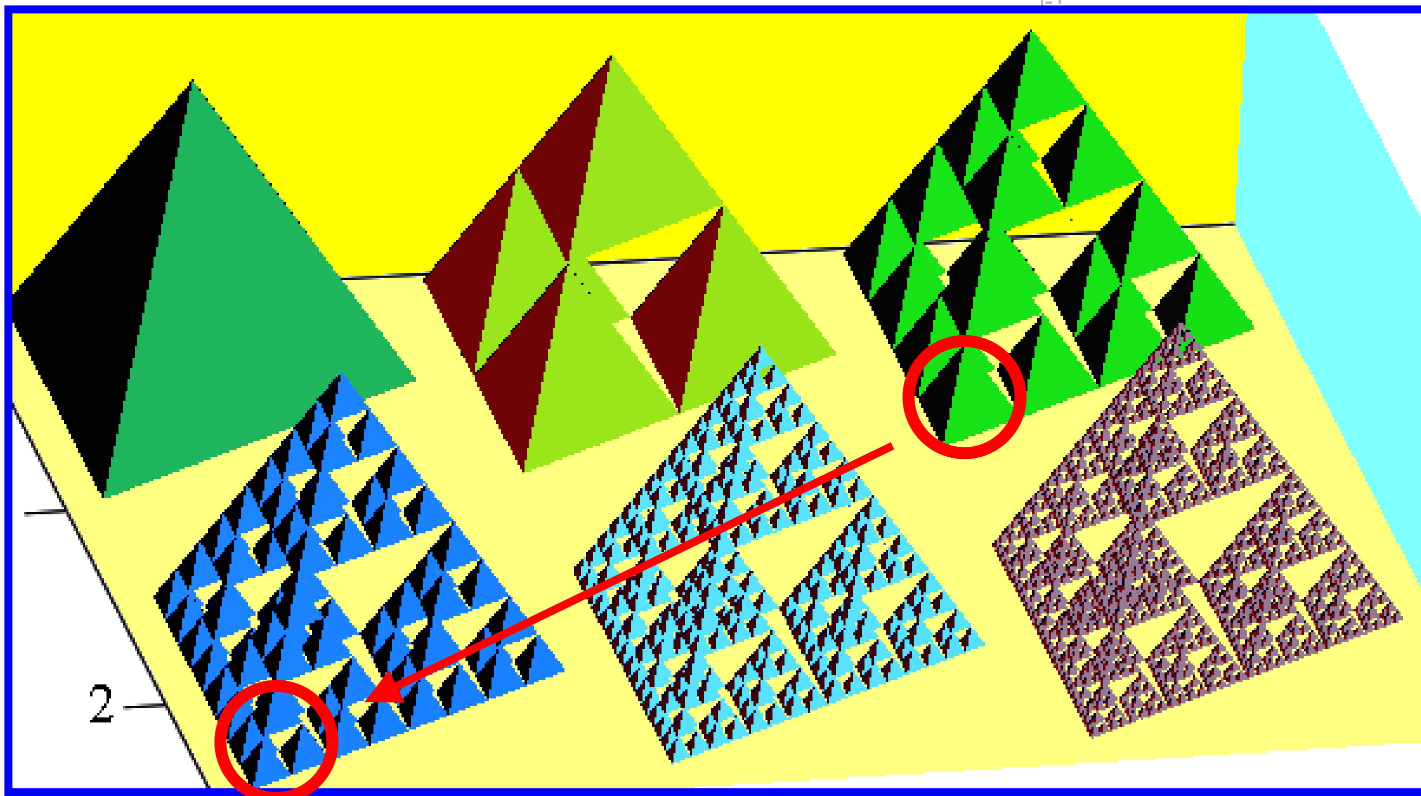
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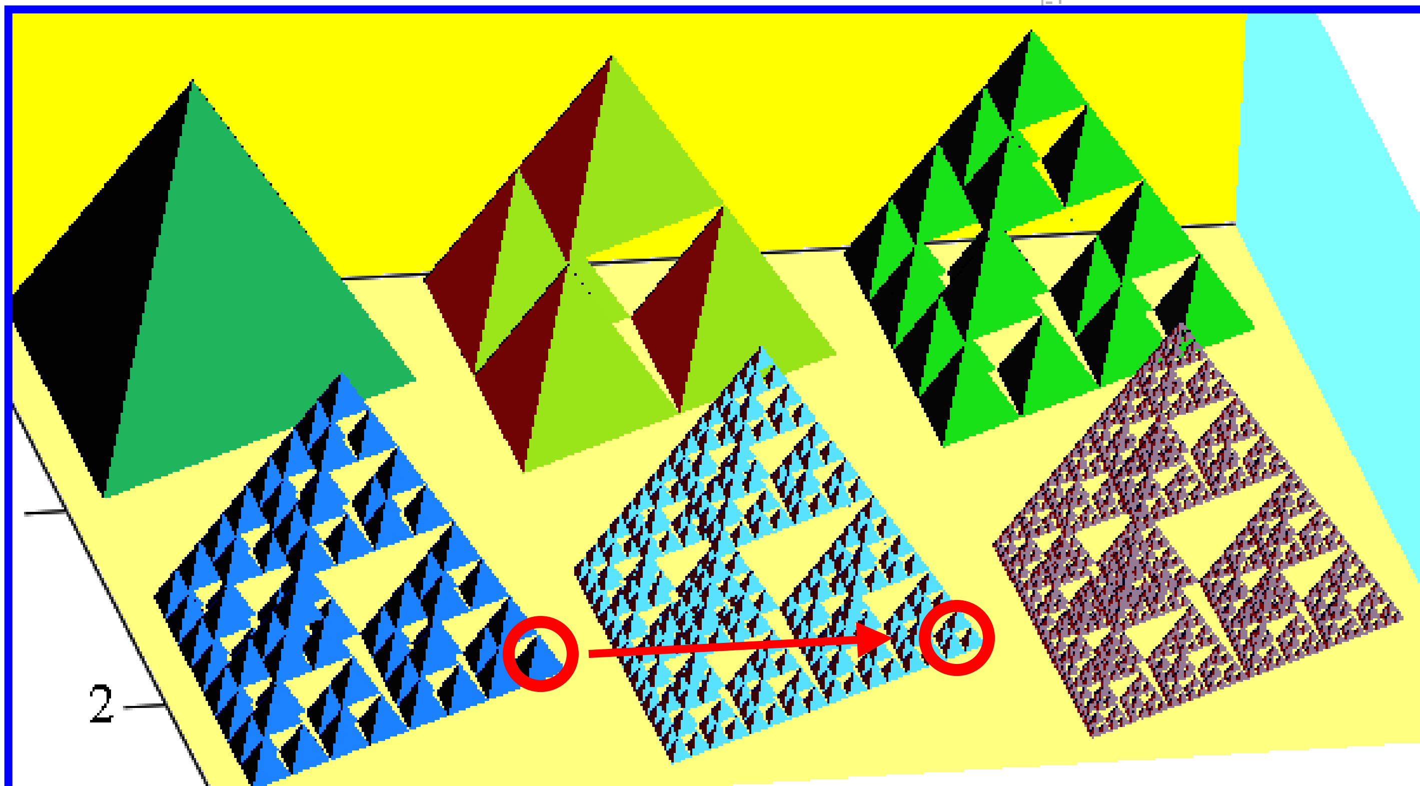
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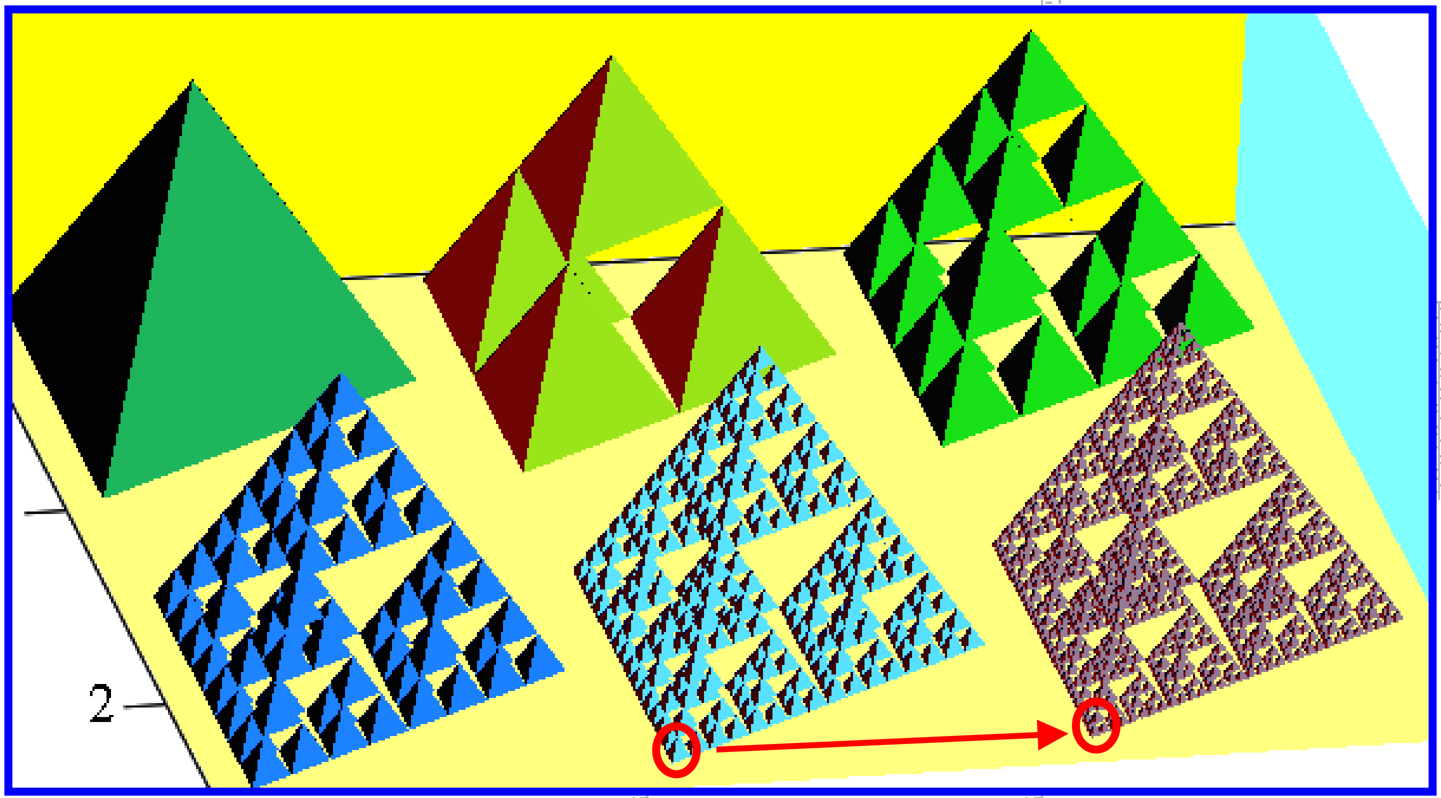
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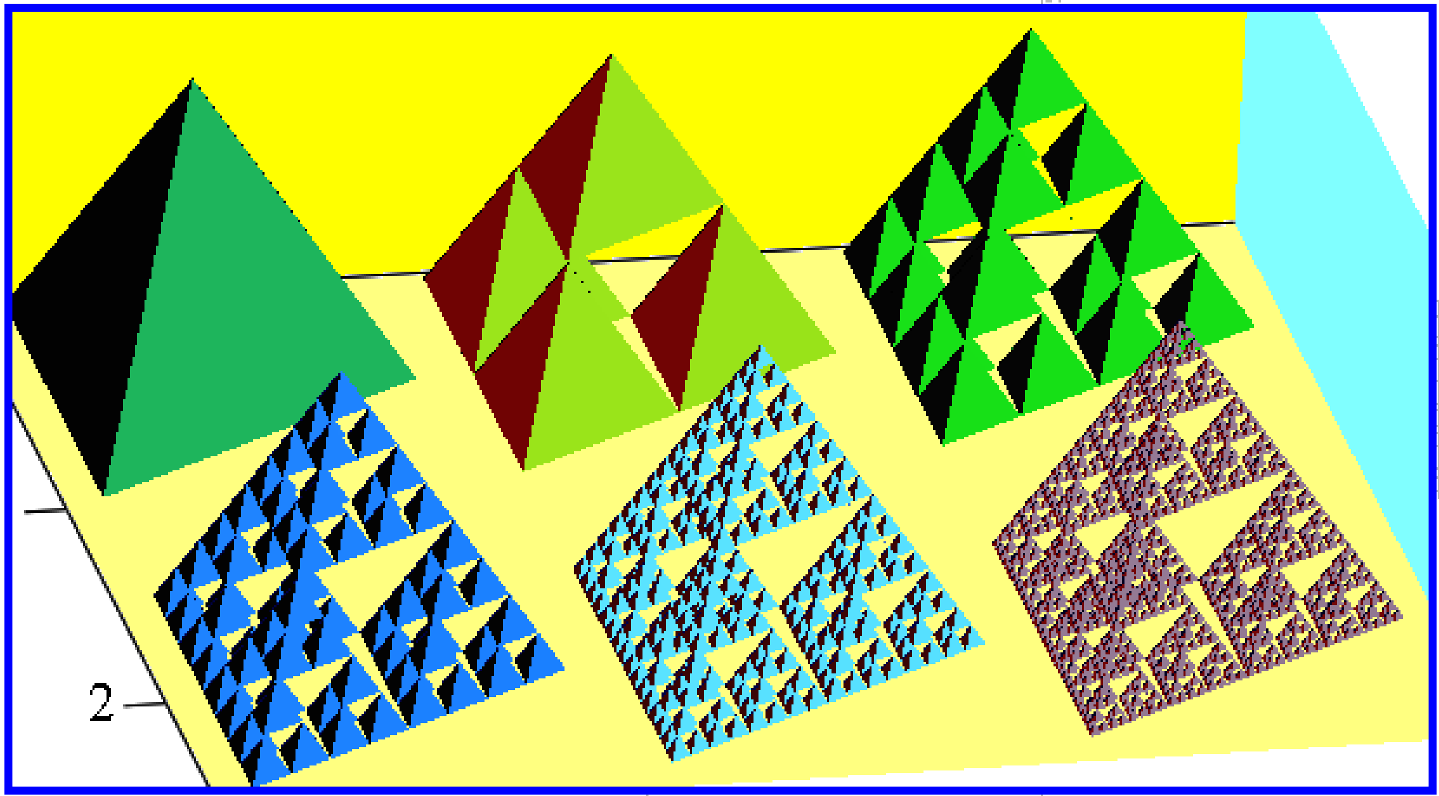


Determine the stage by counting the number of ***sizes*** of openings, the stage-1 has one size of opening, the stage-2 two ***sizes*** of openings, etc…

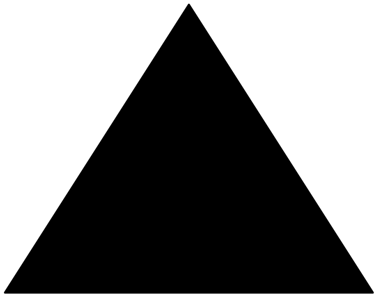
What is happening to all that removed volume in Sierpinski’s tetrahedron? You can view it on my Sierpinski Tetrahedron and its Complement page.

This and more can be seen on on Paul Bourke’s Platonic Solids Fractals and their Complements page.

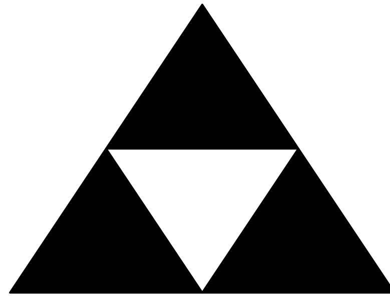
The Sierpinski tetrahedron is a volume analog of

the Sierpinski triangle:

The Sierpinski Triangle: grows in Powers of 3

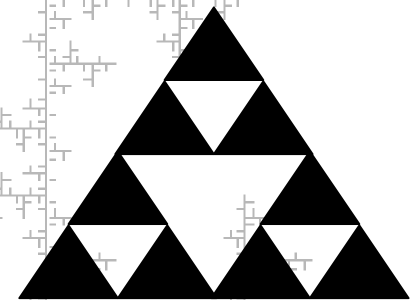


Notice how each triangle ***becomes three triangles*** in the next stage.



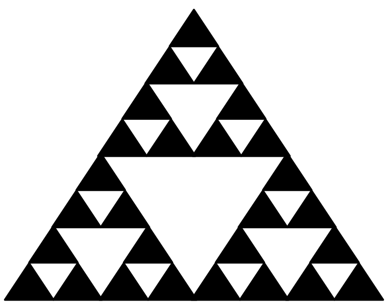
Reduce by ½

Replicate & Rebuild

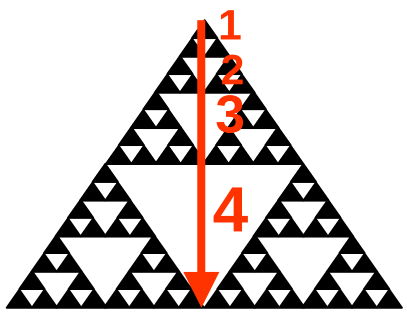


Reduce by ½ again

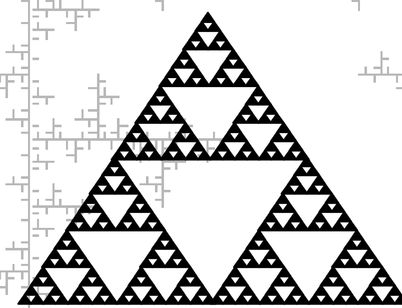
Replicate & Rebuild



With this fractal, it is surface area instead of volume that is decreasing at each stage.



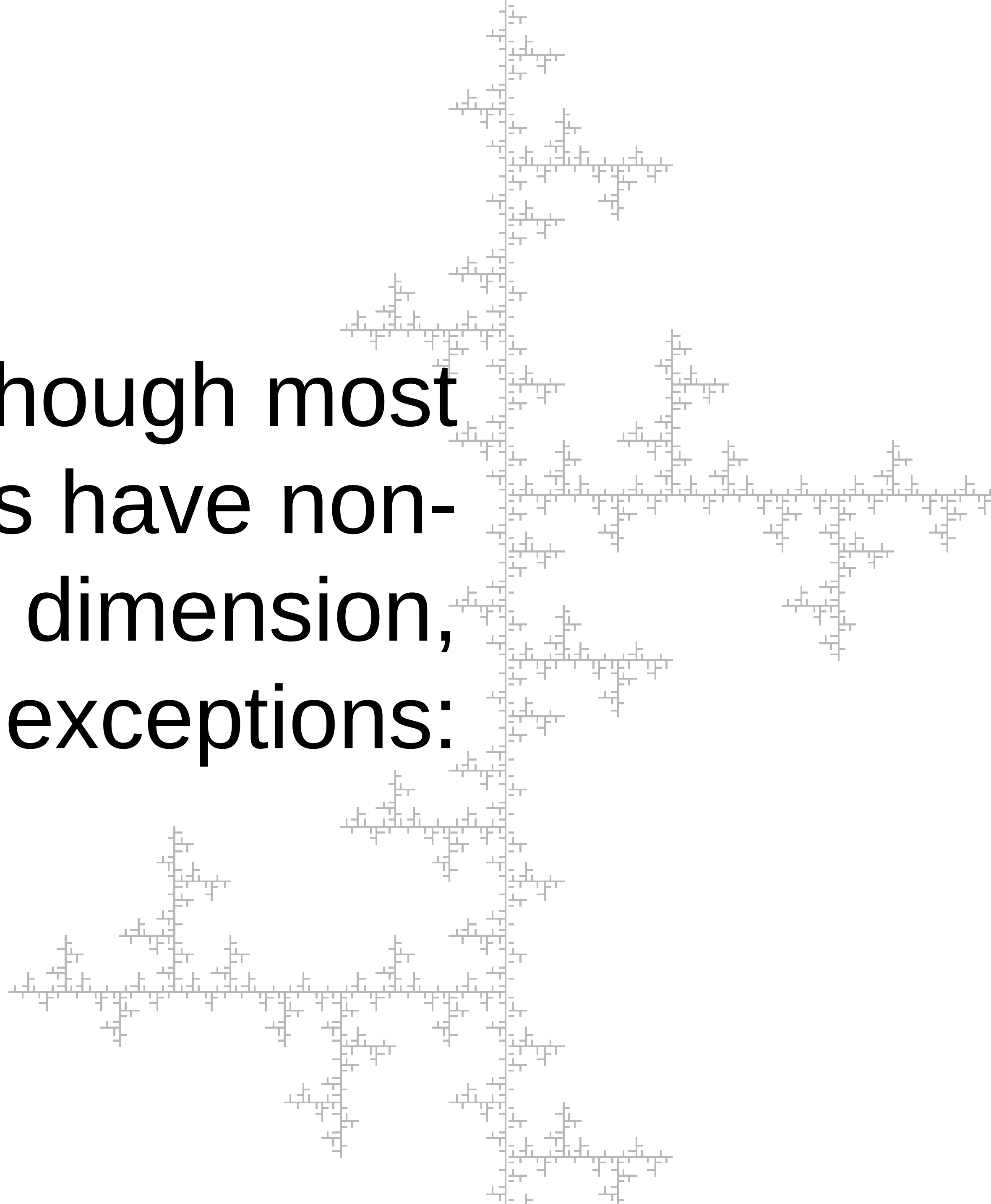
The stage can be determined by the number of different ***sizes*** of openings.



The face of a Sierpinski tetrahedron is a same-stage Sierpinski triangle.

Geometric fractals are typically filling or emptying something, whether it is length, surface area, or volume. The key points are that dimension is: 1) changing, and 2) generally fractional.

Even though most fractals have non-integer dimension, there are exceptions:



For exactly self-similar shapes made of N copies, each scaled by a factor of r, the dimension is

$$\mathrm{Log}(N)/\mathrm{Log}(1/r)$$

The Sierpinski tetrahedron is made of N = 4 copies, each scaled by a factor of r = 1/2, so its dimension is

$$\mathrm{Log}(4)/\mathrm{Log}(2) = 2$$

So the Sierpinski tetrahedron is a shape that is manifestly fractal, but has integer dimension!

Contrast this with the Sierpinski triangle, made of N = 3 copies, each scaled by a factor of r = ½. Its dimension is

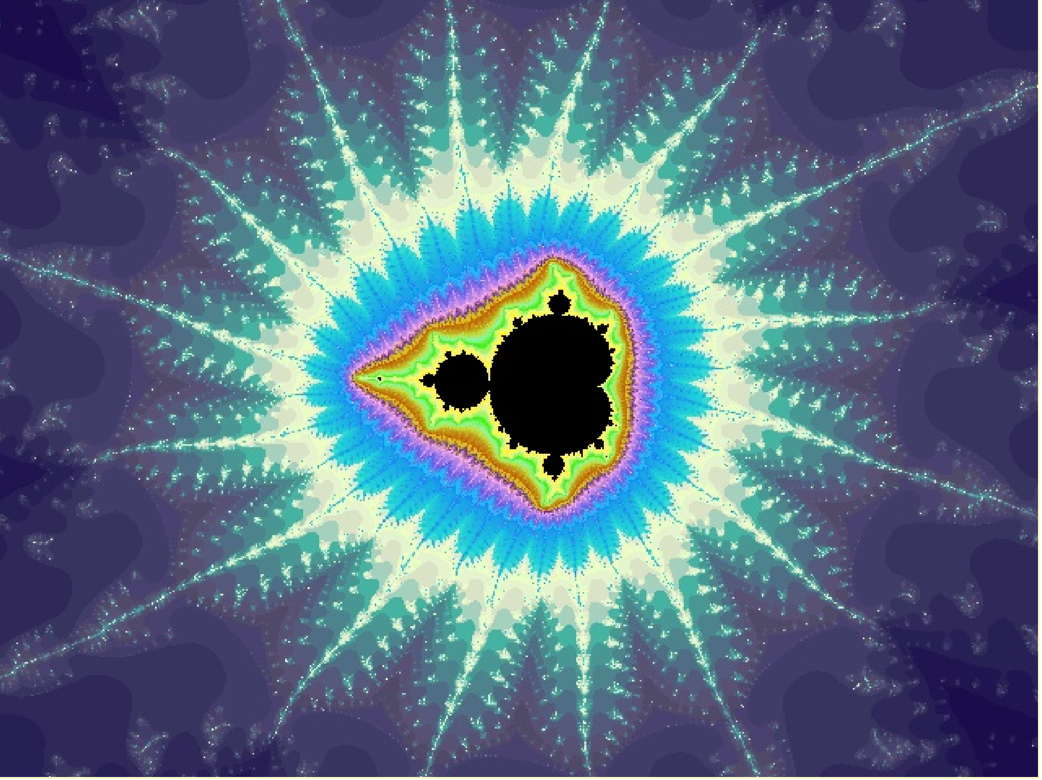
Log(3)/Log(2) ~= 1.58496…..

The Sierpinski triangle has [fractional dimension](), more typical of fractals.

The [exact answer]() above is Log(3)/Log(2). The approximate answer is the decimal approximation 1.58496… Rule of thumb: keep answers in exact form unless a decimal approximation is requested, and when requested, wait until the very end to convert to a decimal to avoid rounding error.

Self-similarity: this is a big idea, and it only truly applies to geometric fractals; however, it is used as a concept to talk about *all types of fractals*.

Something is self-similar when every little part looks exactly like the whole. The only place this can really happen is in a perfect (Ideal) system at infinity; however, in order to speak about fractals generally, one must embrace the ***concept*** of self-similarity in a broad way.



(Everything on this slide links to relevant websites; url's are included since links aren't coming thru the conversion to PDF.)

Chaos

http://classes.yale.edu/fractals/Chaos/ChaosIntro/ChaosIntro.html

Multifractals

http://classes.yale.edu/fractals/MultiFractals/welcome.html

Random Fractals

http://classes.yale.edu/fractals/RandFrac/welcome.html

Complex Fractals

http://www.geom.uiuc.edu/~zietlow/defp1.html

Mandelbrot discusses fractals

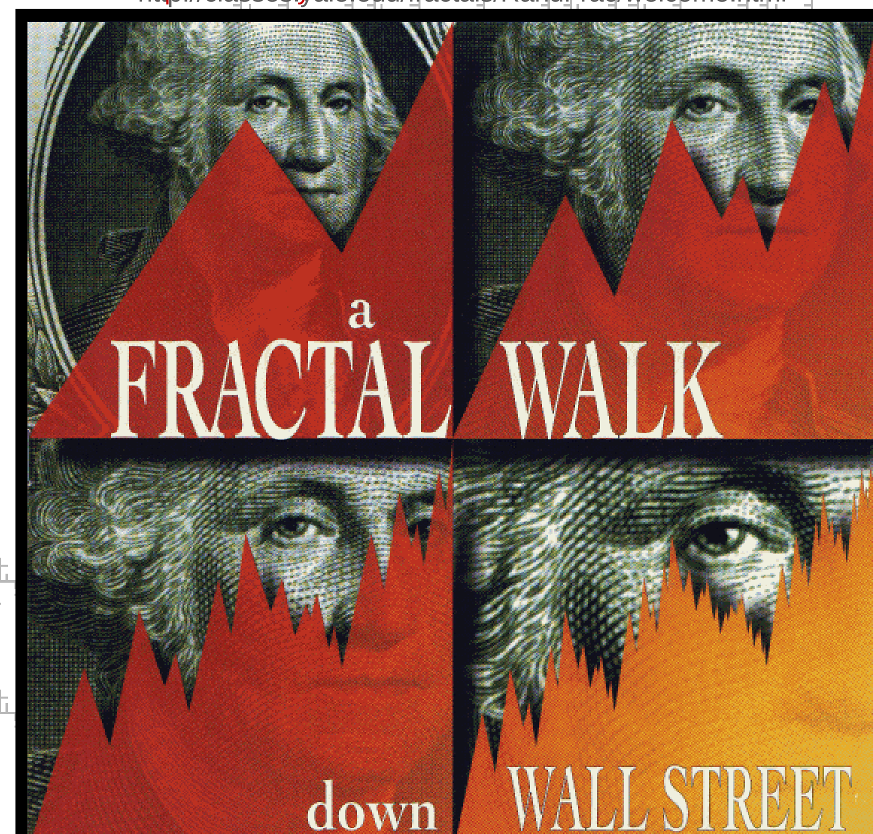
http://www.yale.edu/opa/v31.n20/story6.html

Mandelbrot Set

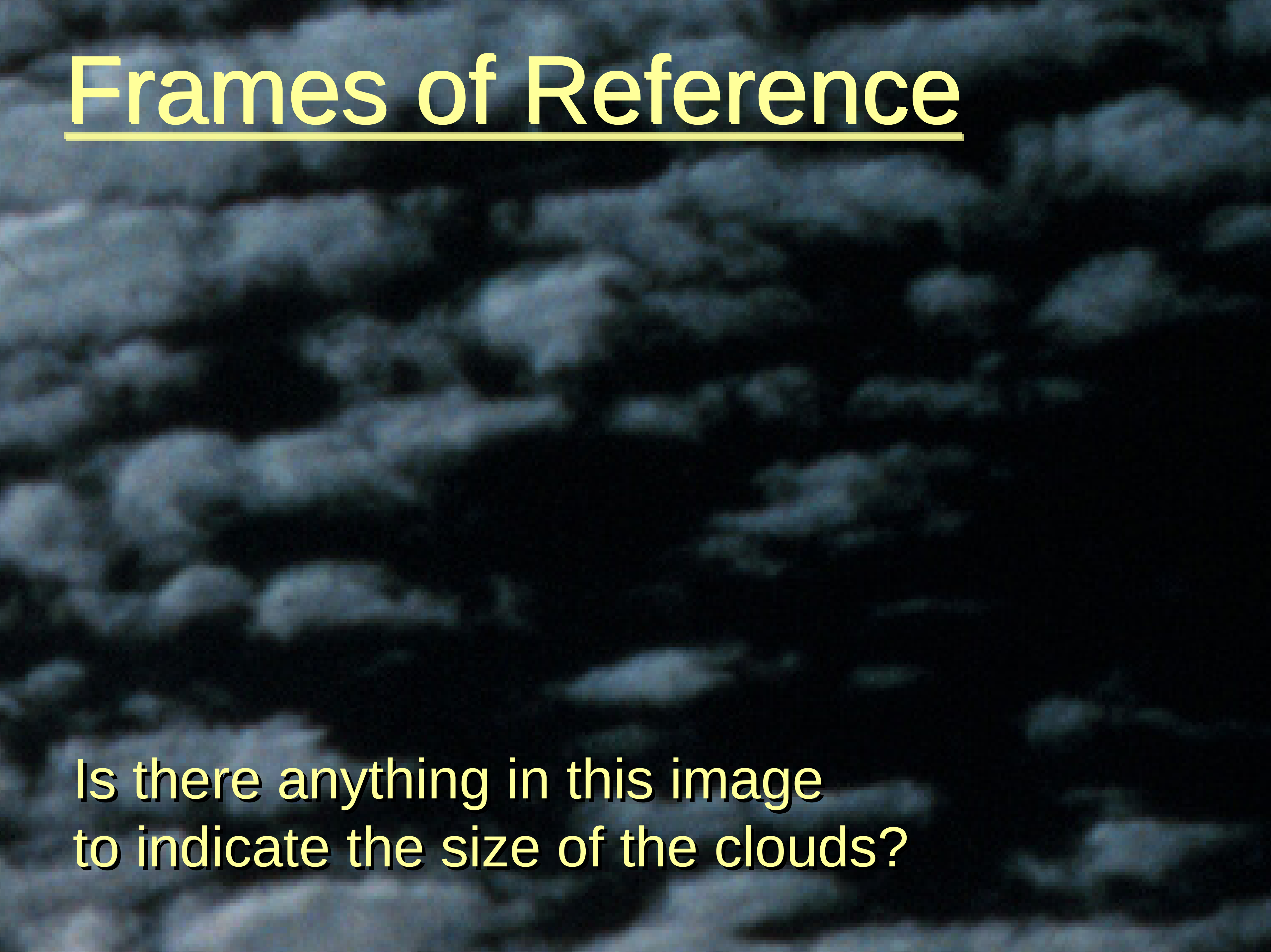
http://www.ddewey.net/mandelbrot/noad.html

Julia Sets

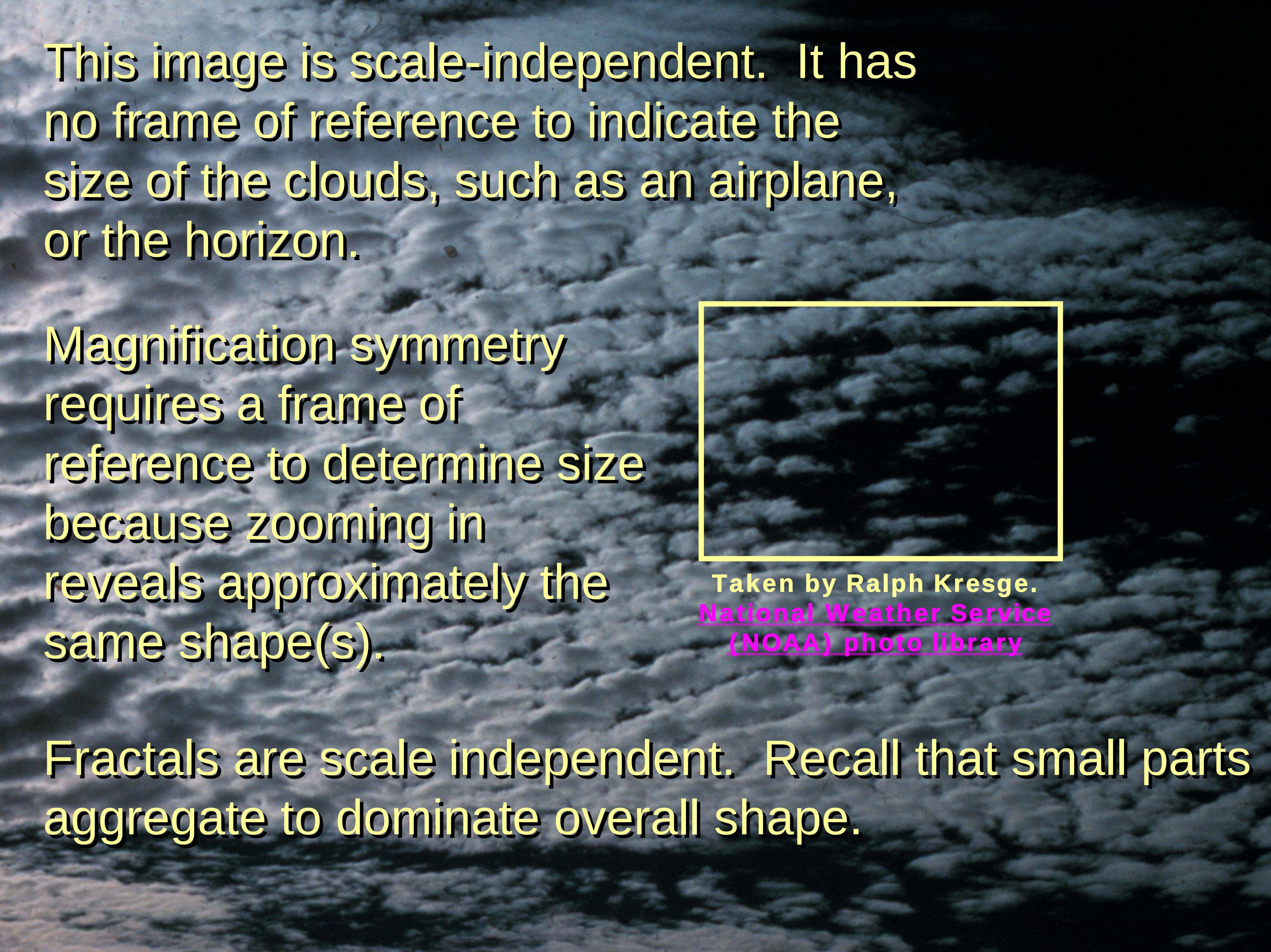
http://www.ibiblio.org/e-notes/MSet/Period.htm



Frames of Reference



Is there anything in this image
to indicate the size of the clouds?



This image is scale-independent. It has no frame of reference to indicate the size of the clouds, such as an airplane, or the horizon.

Magnification symmetry requires a frame of reference to determine size because zooming in reveals approximately the same shape(s).

Taken by Ralph Kresge.
National Weather Service (NOAA) photo library

Fractals are scale independent. Recall that small parts aggregate to dominate overall shape.

Within a fractal system, the smallest scale is present in multitudinous numbers. The medium scale has a significant presence, with a comparative handful of giants.

We see examples of this in **bugs** and **galaxies**, also in stars within galaxies. The small are proliferate while the huge are few and far between.

Another image by Ralph Kresge of the NOAA. This time a tree provides a frame of reference, and it looks like some birds are flying around. Visit one of their clouds photo libraries.

Examine exponential growth in a geometric fractal: the Menger Sponge.

The Menger Sponge is part of a series of fractals, in that while it is Volumetric, it has Length and Area analogs. The Area analog, the Sierpinski Carpet (seen in image), is used by Fractal Antenna Systems as an antenna in cell phones. The number of scales allows for a wide range of receptions.

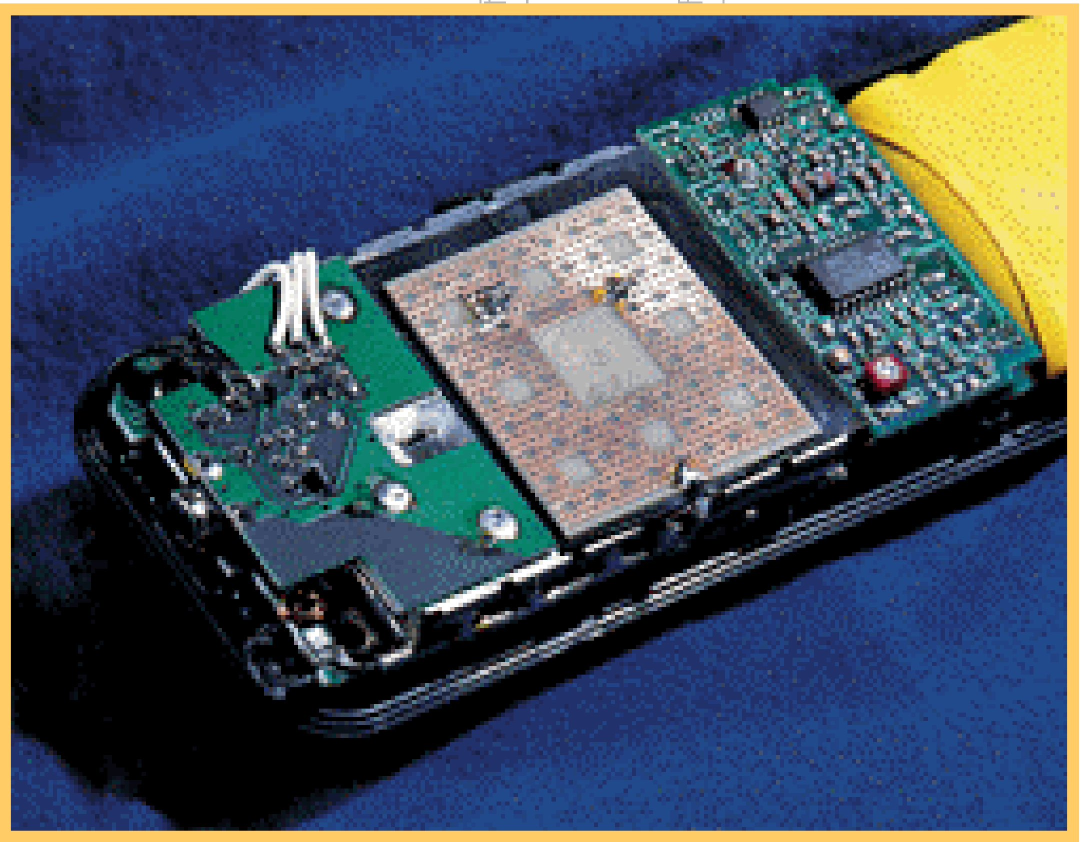
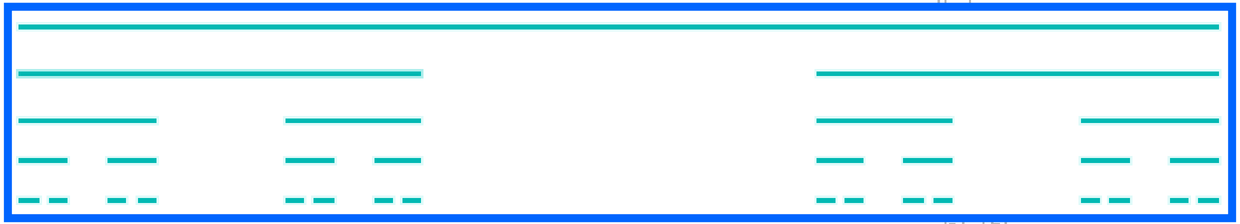
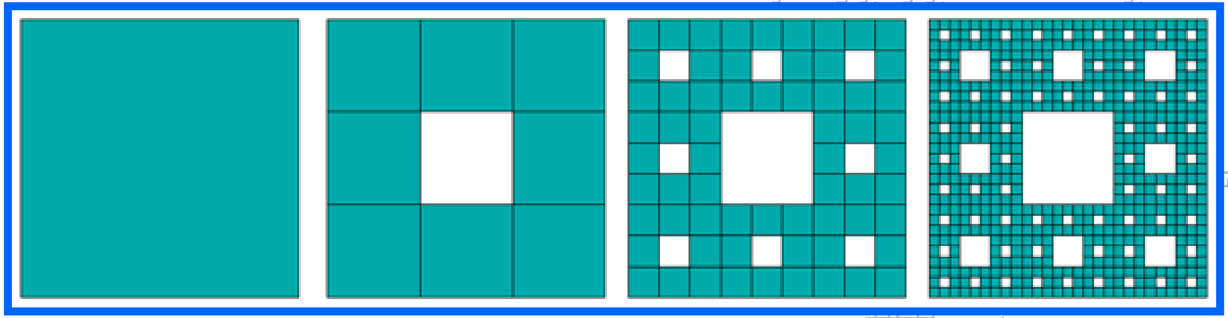


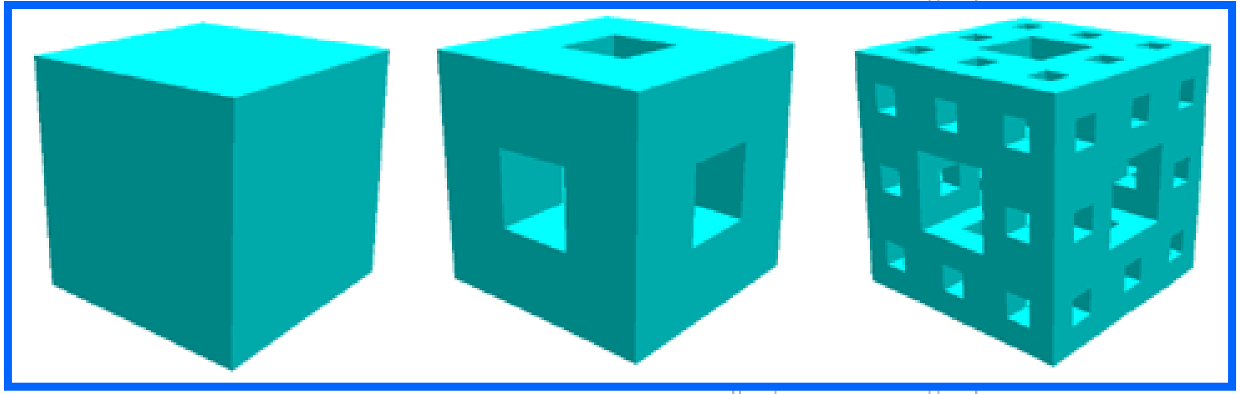
Image from Yale's Fractal Antennas page



Length analog, the Cantor Set

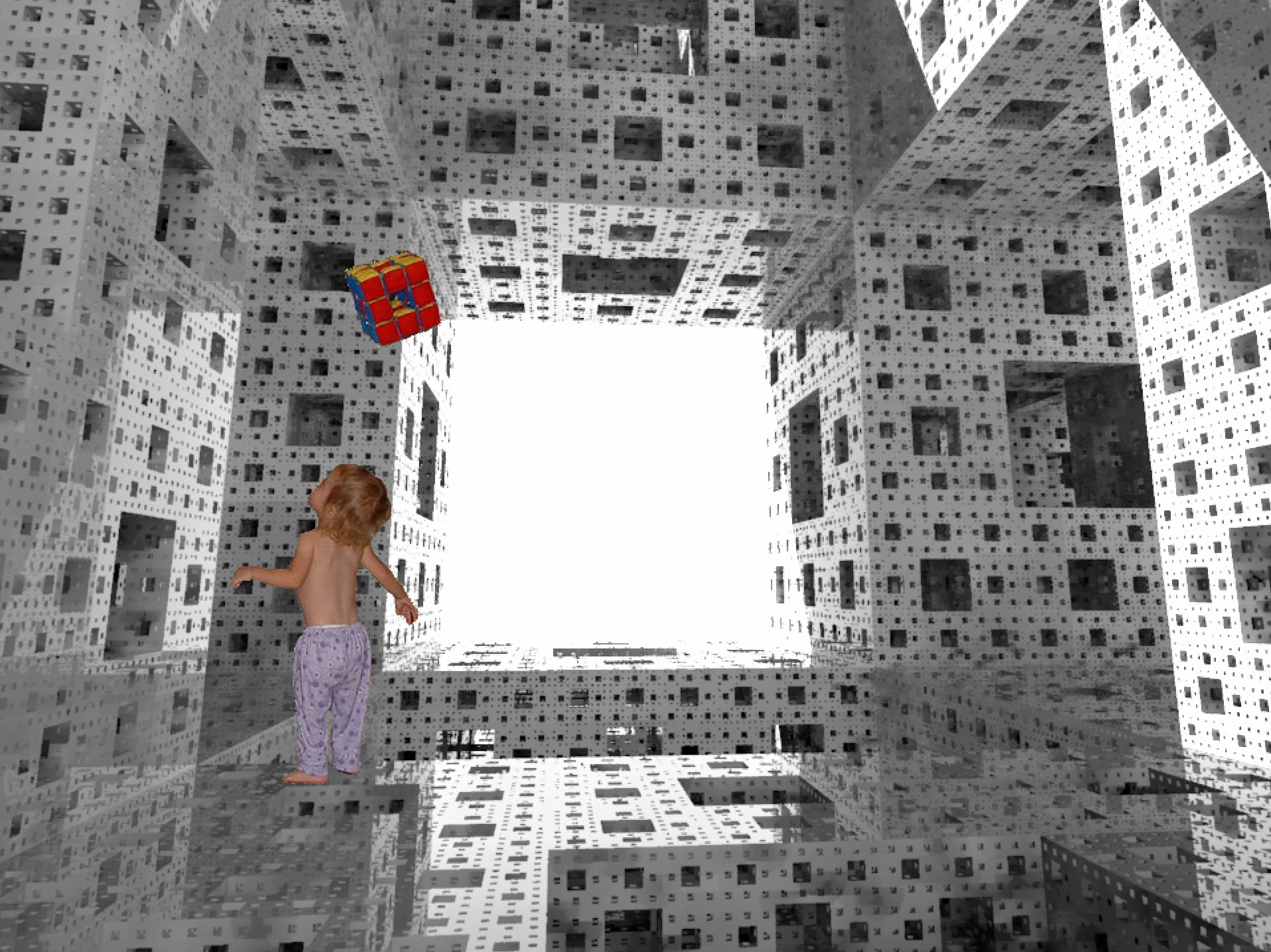


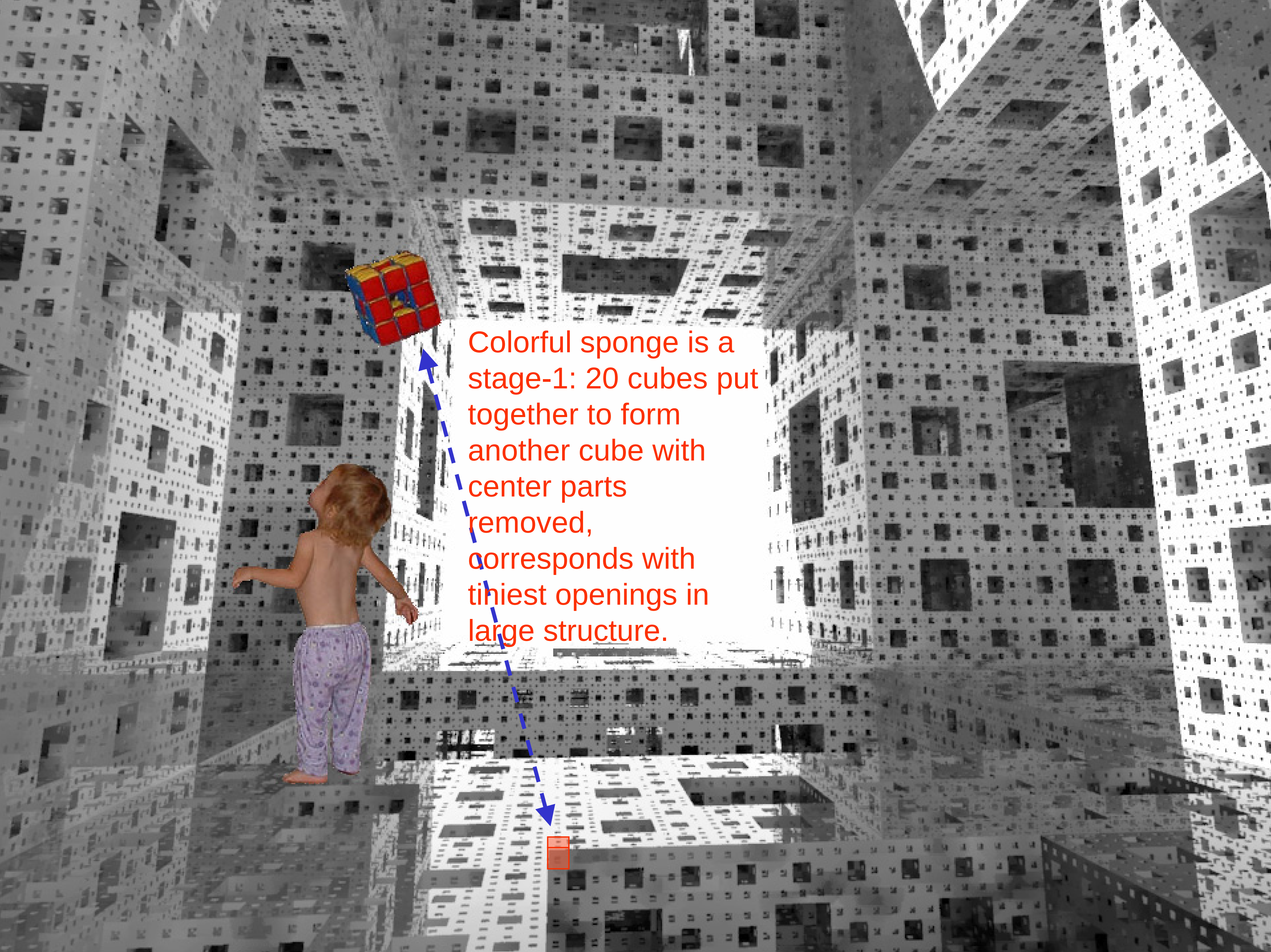
Area analog, the Sierpinski Carpet

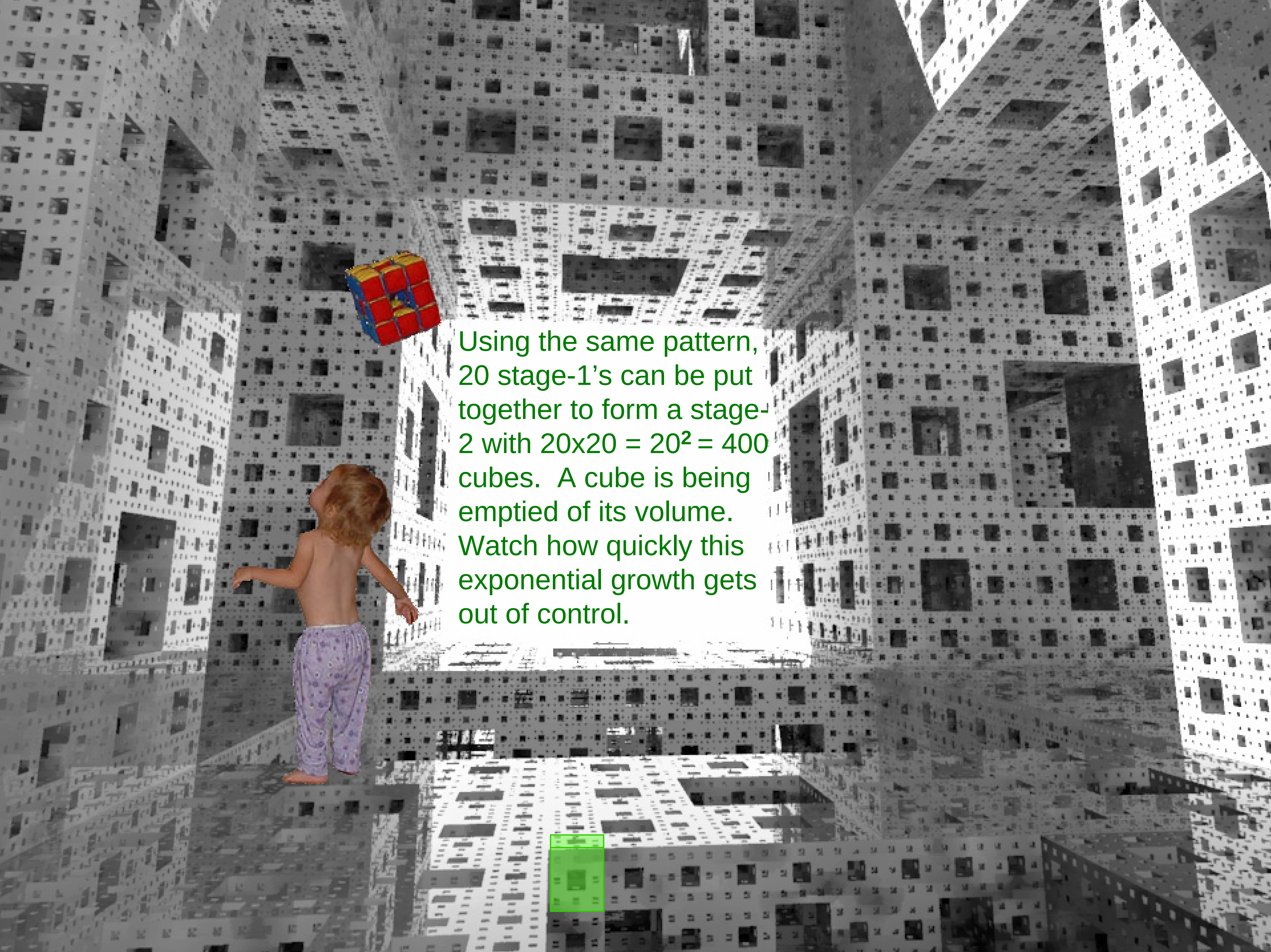


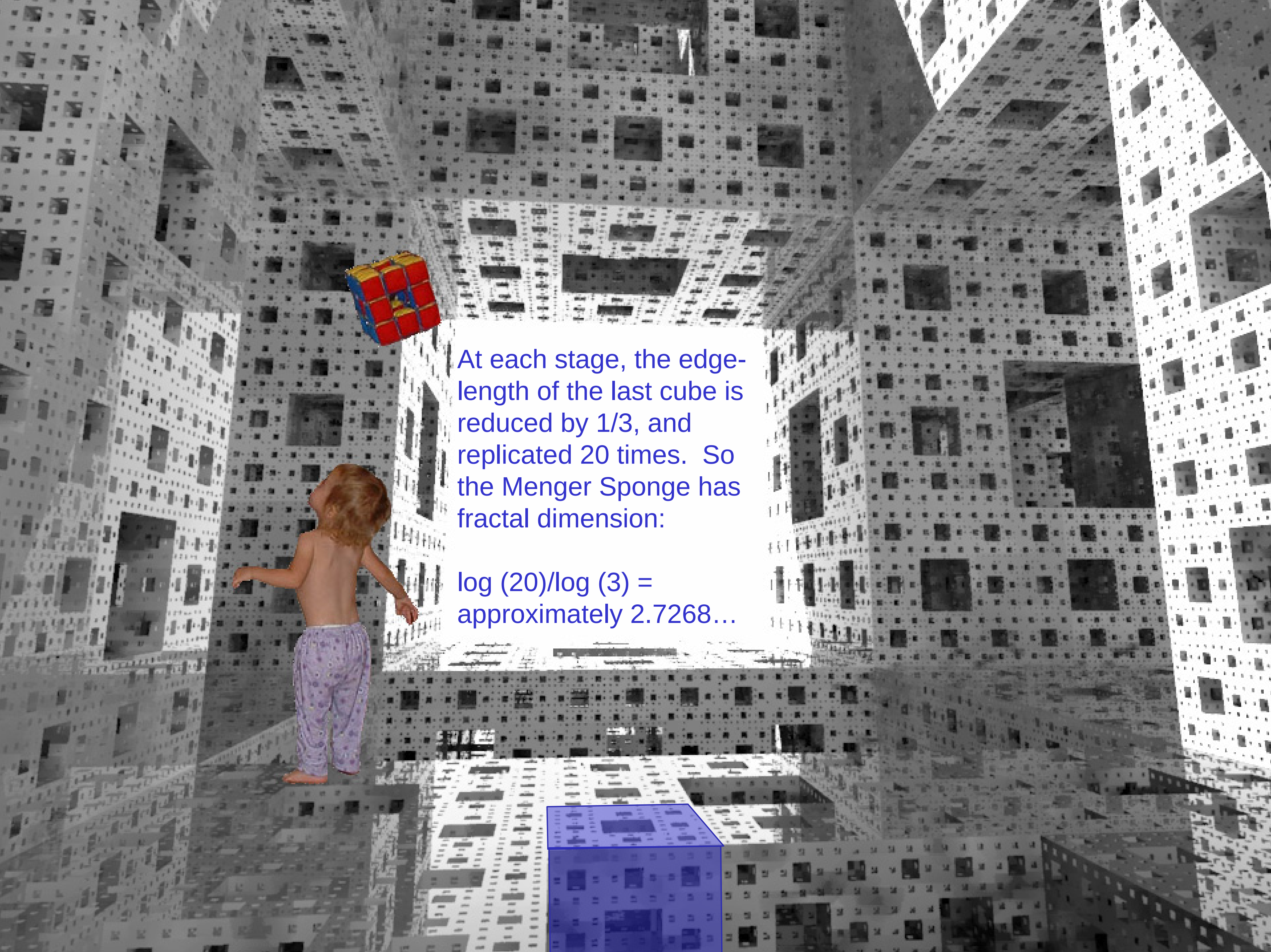
Volume analog, the Menger Sponge

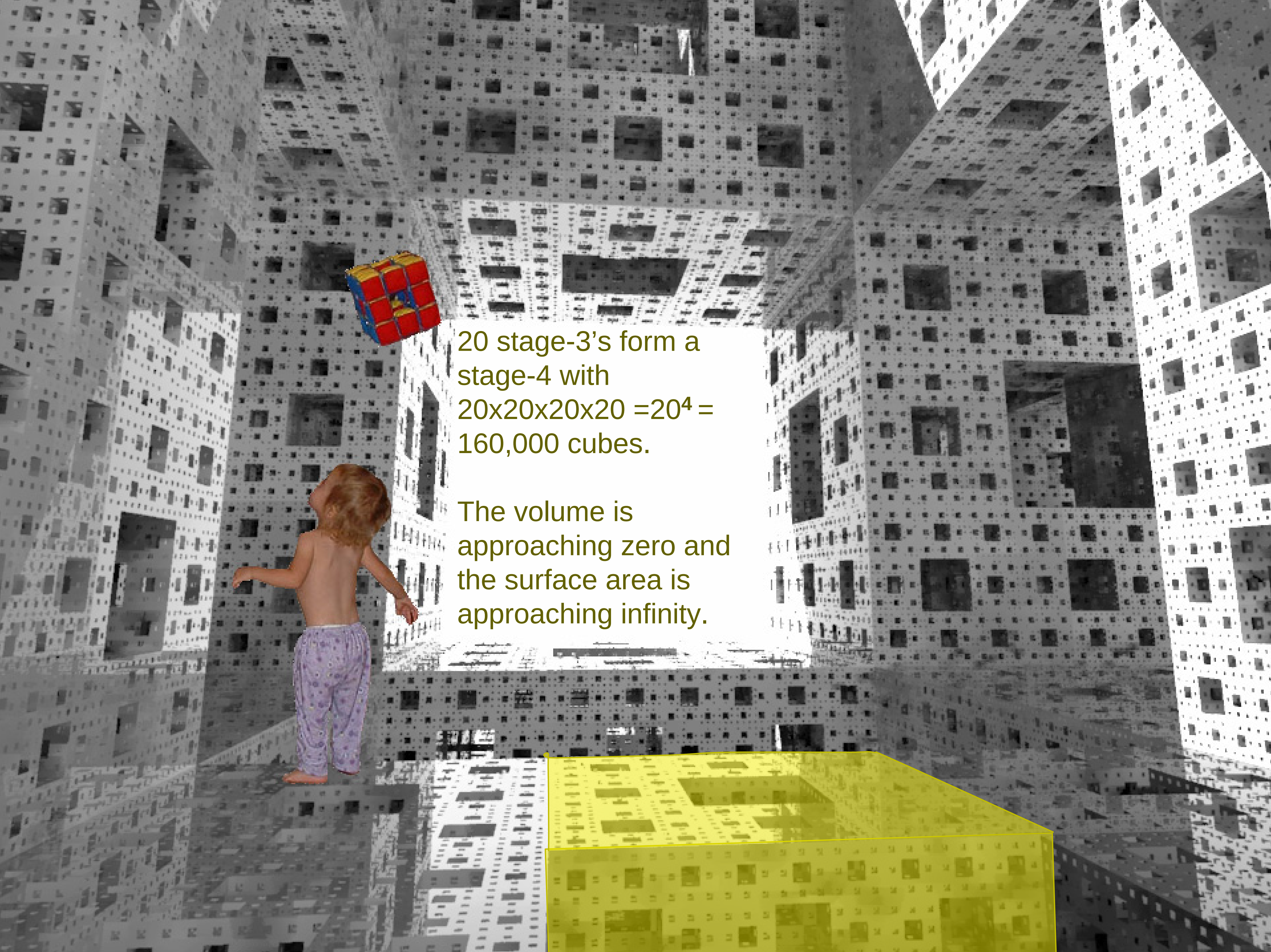
Images courtesy of Paul Bourke

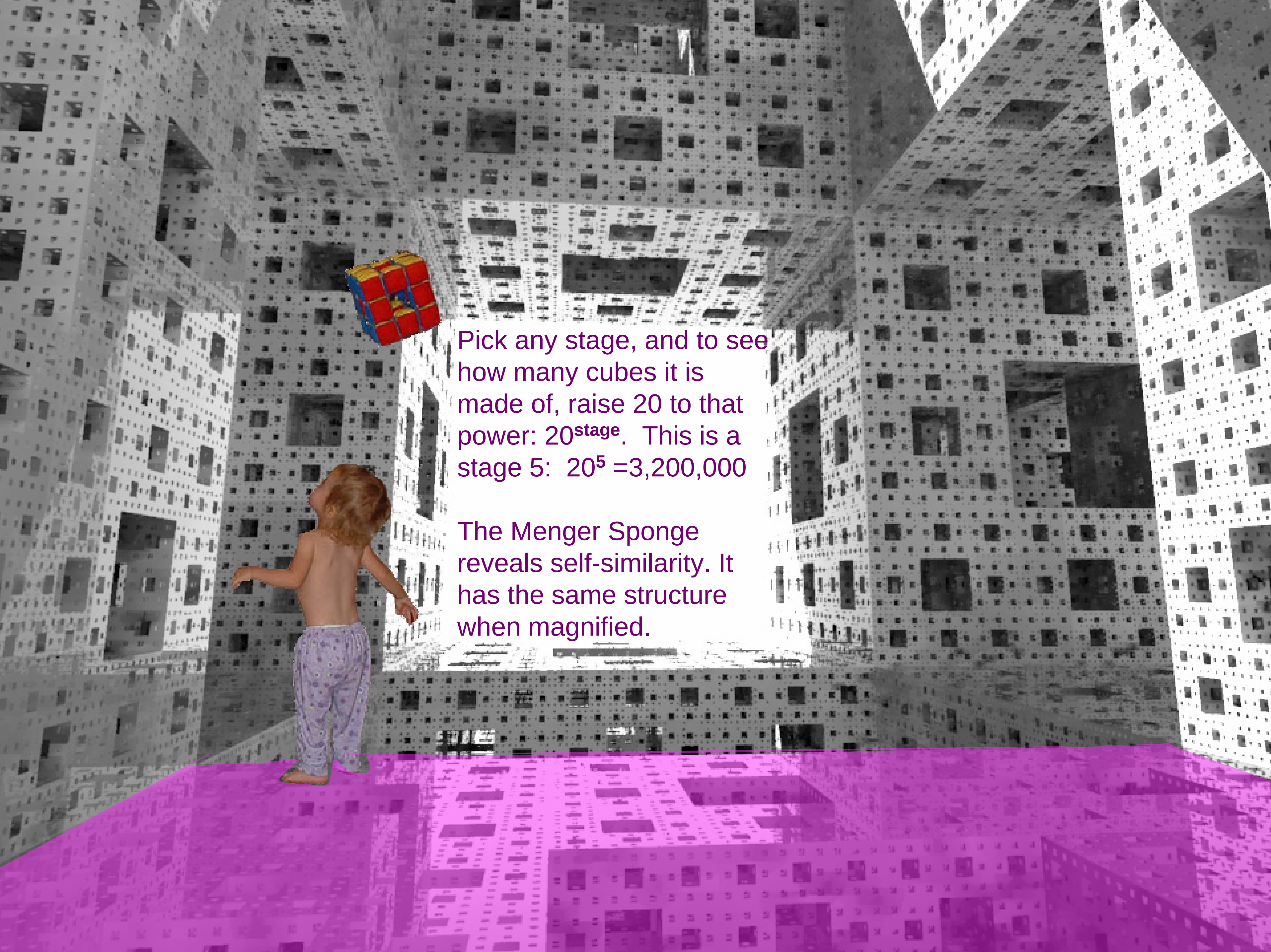



Colorful sponge is a stage-1: 20 cubes put together to form another cube with center parts removed, corresponds with tiniest openings in large structure.


Using the same pattern, 20 stage-1's can be put together to form a stage-2 with 20x20 = 20^2 = 400 cubes. A cube is being emptied of its volume. Watch how quickly this exponential growth gets out of control.


At each stage, the edge-length of the last cube is reduced by 1/3, and replicated 20 times. So the Menger Sponge has fractal dimension:
log (20)/log (3) = approximately 2.7268…


20 stage-3's form a stage-4 with 20x20x20x20 =20^4 = 160,000 cubes.
The volume is approaching zero and the surface area is approaching infinity.



Pick any stage, and to see how many cubes it is made of, raise 20 to that power: 20^{stage}. This is a stage 5: 20^5 =3,200,000

The Menger Sponge reveals self-similarity. It has the same structure when magnified.

How far can this go?
As far as you want it to.
There is no reason to stop here.

This is a stage-6. It is made up of 20^6 = 64 million cubes.

There is no uncertainty about the way it will grow or what it will look like after any number of stages of growth.

Image courtesy of Paul Bourke.

Fractals Across the Disciplines

A selection of topics from the **Yale Fractal Geometry** web page **A Panorama of Fractals and Their Uses**:

- Art & Nature
- Architecture
- Astronomy
- Finance
- History
- Industry
- Literature
- Music
- Nature & Fractals
- Physiology
- Poetry
- Psychology
- Social Sciences

(The categories all link to their respective pages.)

Recapping the main fractal theme addressed in this presentation:

Fractals operate under a Symmetry of Magnification (called Dilatation or Dilation in literature). Different types of fractals share a common ground of parts that are similar to the whole. Even though self-similar substructure must technically be present all the way to infinity for something to be called fractal, the concept of fractility is loosened to apply to forms (esp. natural) with only a handful of levels of substructure present.

The simplification of complexity leading to useful results that we have been looking at is not unique to the field of fractals, it is a theme that runs throughout mathematics, although the methods of simplification vary.

> “Mathematics is about making clean “simplified” concepts out of things that we notice in the world around us. In the world [staying with fractals as an example], when it is applicable we make a clean concept by assuming the existence of self-similarity—infinite levels of substructure—when there are only a few, pushing beyond reality.” *(Priscilla Greenwood, Statistician and Mathematical Biologist at Arizona State University)*

Math “works” because these simplified systems “work”. Mathematicians could well be called The Great Simplifiers.