COMPUTER RECREATIONS
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COMPUTER RECREATIONS

Probing the strange attractions of chaos

by A. K. Dewdney

Chaos has strange attractions for the mind that can see patterns therein. Some physical systems that exhibit chaotic behavior do so because they are in a sense attracted to such patterns. As a bonus, the patterns themselves are strangely attractive. Some readers may already be aware that the geometric forms underlying chaos are called strange, or chaotic, attractors [see "Chaos," by James P. Crutchfield, J. Doyne Farmer, Norman H. Packard and Robert S. Shaw; SCIENTIFIC AMERICAN, December, 1986]. Strange attractors can be generated with a home computer.

Before setting off with me, readers must be equipped with a protective coating of physical intuition. In particular, what is an attractor? Roughly speaking, an attractor is a generalization of the notion of equilibrium; an attractor is what the behavior of a system settles down to, or is attracted to. The pendulum is a simple physical system that illustrates the concept of an attractor. Suppose an ordinary pendulum moves under frictional forces that slow it eventually to a standstill. One can describe the pendulum's motion by means of a so-called phase, or state, diagram in which the angle the pendulum makes with the vertical is graphed against the rate at which the angle changes. The swinging motion of the pendulum is represented by a point circling the origin in the phase diagram; as the pendulum loses energy, the point spirals into the origin, where it ultimately comes to rest. In this case the origin is called an attractor because it seems to attract the moving point in the phase diagram. Readers would be correct in thinking there is nothing strange about an attractor consisting of a single point.

A slightly more complicated attractor underlies the motion of a grandfather clock. Here an escapement mechanism feeds energy to a pendulum to keep it from slowing down. If one starts the clock with an overly energetic push of the pendulum, it slows down to the tempo prescribed by the escapement but thereafter slows no further. If the clock is started with a push that is too gentle, however, the pendulum behaves like an ordinary one: it slows to a standstill as before. In the case of the overly energetic push, the pendulum's motion in a phase diagram is a spiral that winds ever more tightly about a circular orbit. Here the attractor is a circular loop. In this context a circle is no stranger than a point.

An ordinary pendulum can be made to show chaotic behavior by introducing a vertical, vibratory motion: if the point of support is moved up and down in a sinusoidal manner by an electric motor, the pendulum may begin to swing crazily, exhibiting no evidence of periodic behavior whatever.

To introduce chaos, however, I have selected a different physical system. Imagine an arrangement of three amplifiers in which the first amplifier outputs a signal x that is fed to the other two. The second amplifier outputs the signal $1 - x$ in response to x. The third amplifier takes the two signals, x and $1 - x$, as input. It generates the product, $x(1 - x)$, of the two signals and feeds it back to the first amplifier, which also receives a control voltage, r, as input. One additional component, a device that samples its input and delivers the same voltage as output for a short time, completes the circuit; it is inserted in the output line from the first amplifier. The three-amplifier circuit does a voltage dance that becomes more hectic as the control voltage r is gradually increased.

When r is less than 3 and x initially has some nonzero setting, the circuit oscillates briefly before it settles down to a specific value of x that remains the same thereafter. This value constitutes a single point attractor. If the control voltage, r, is now raised to a level just above 3, the circuit flutters between two values of x. At this level of r the circuit is said to be bistable and the attractor consists of two points. As r is increased further, the circuit oscillates among four points; yet another increase yields an eight-point attractor. The pattern of doubling and redoubling goes on as the knob controlling r is turned to higher values, until at a setting roughly midway between 3 and 4 the circuit suddenly goes crazy. It hunts endlessly at electronic speed for the simple recurring patterns that marked its earlier existence. Its behavior is now governed by a strange attractor that has a potential infinity of values. The result is chaos.

Electronically literate readers may be tempted to build such a circuit. Others may simulate it on a computer of any size, viewing the dance with great clarity on the display screen. To do so, they merely need to write a simple program that computes the iterated equation $x \leftarrow rx(1 - x)$. The program, which I call CHAOS1, has a core that consists of six instructions:

```
x ← .3
for i ← 1 to 200
    x ← rx(1 − x)
for i ← 1 to 300
    x ← rx(1 − x)
      plot (200x, 100)
```

The variable x starts at the value .3. CHAOS1 then enters a loop that iterates the basic equation 200 times to allow transients to die away. The transients are inherent in the equation itself, not in imprecise arithmetic. The reasons for this will be made clear in geometric terms below. The program then enters a new loop that iterates the equation 300 more times, plotting the value of x on each occasion.

The number 100 used in the plot instruction above is more generic than specific; here the screen has dimensions 200 by 200. The horizontal coordinate, $200x$, spreads the various values computed for x (always between 0 and 1) across one row of the screen, which is set at a height of 100—halfway up the hypothetical screen.

Depending on the setting for the control variable r, the core program will either plot a single point 300 times or several points fewer than 300 times each. It may even try to capture chaos by plotting 300 different points of a strange attractor. If the iteration limit is increased, more of the strange attractor will be seen. In all cases, once the iteration process has settled down, the x values jump in a systematic way from one point of the attractor to another. The attractors are also called orbits, regardless of whether they have a finite or an infinite number of points.

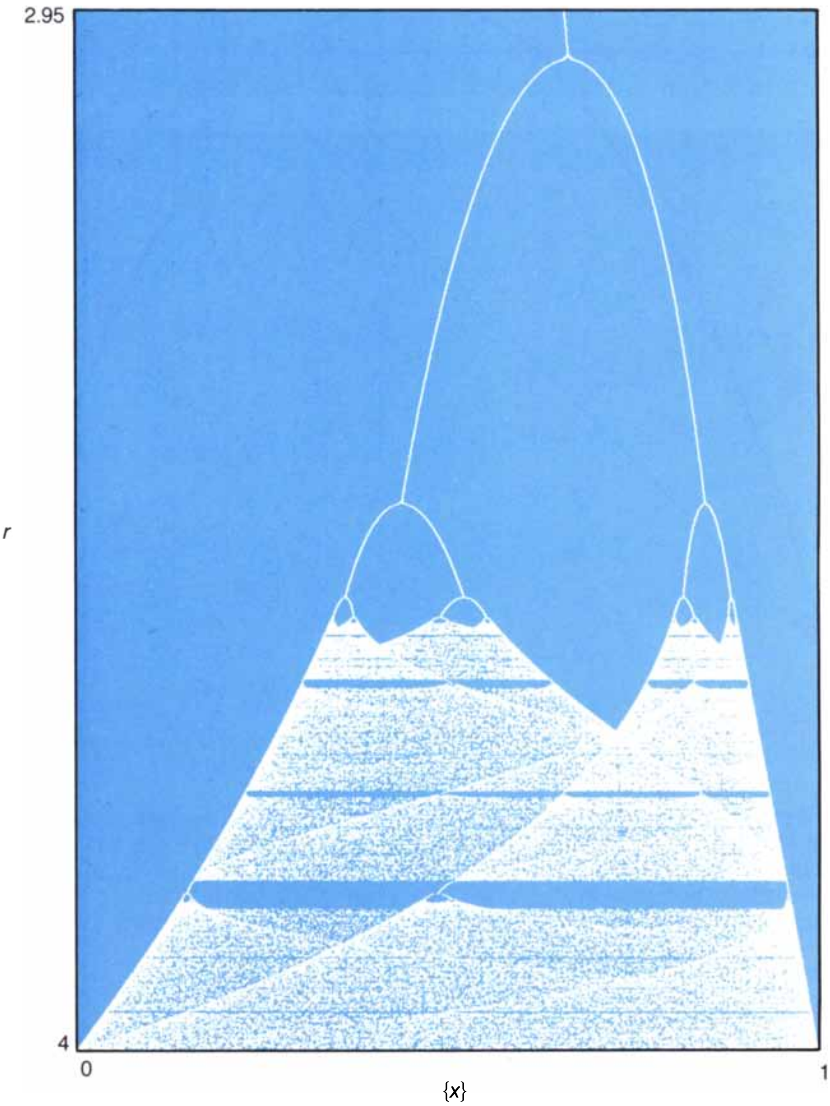
A complete picture of the behavior of the simple amplifier circuit emerges if the program computes a raft of plots, each plot below the last one [*see illustration at right*]. The plots result from a succession of r values that run from 2.9 to 4.0 in, say, 200 steps from the top of the screen to the bottom. A more elegant picture emerges if more steps are used, say 4,000, but in this case the diagram will not fit on the screen and it must be plotted to be seen as a whole.

For values of r less than 3.56 (the more precise value is 3.56994571869) the attractors of the simple dynamical system embodied in the iterated equation $x \leftarrow rx(1-x)$ consist of a few points. These points, which represent nonchaotic behavior, are arranged in three large bands and an infinity of smaller ones. The attractors become strange as r approaches 3.56. Here chaos sets in as the hitherto smoothly bifurcating lines suddenly fall into a pepper-and-salt madness. Strangely enough, the chaotic regime vanishes from time to time as r continues its inexorable march to 4.

The entire plot is called a bifurcation diagram. When it is viewed sideways, it resembles the spectrum of chaos from a star named x. The plot is embellished by curves and attractively shaded folds. The reasons for the ornamental details are mysteries that can be explained only by the theory of chaos. I shall delve more into that topic below. For the present there is a mystery closer to home in the minds of most readers: Why does the innocent-looking equation behave so strangely?

The equation's behavior for nonchaotic values of r can be simulated geometrically by drawing a parabola described by the equation $y = rx(1-x)$, where x is the horizontal variable and y is the vertical variable. Now superpose on the parabola the diagonal line $y = x$. Such a procedure has been followed in the top illustration on the next page, where r has been set at 3.3, a value at which the system's attractor consists of two points. To show how the system behaves, an initial value of x is chosen. I have picked .3, although almost any other value will do as well.

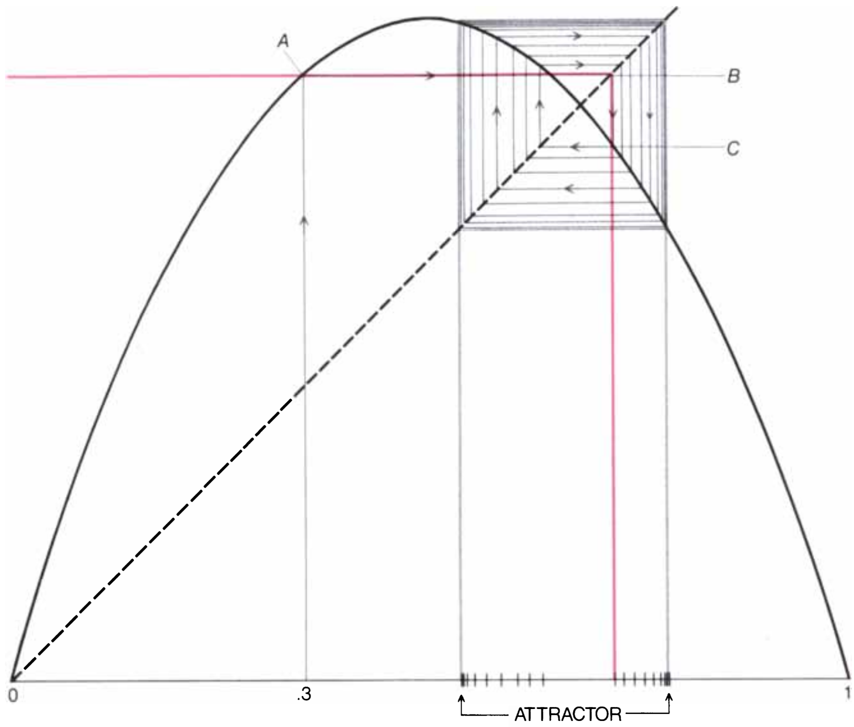
The first iteration of the equation is simulated by drawing a vertical line beginning at the point $x = .3$ at the bottom of the graph and continuing it upward until it hits the parabola. I have labeled the point where the line hits the parabola as A. The height of the intersection determines the corresponding value of y. In the second iteration that value of y is fed back into the equation as the x variable. Graphically the procedure corresponds to measuring the height of the intersection, marking it on the horizontal axis and drawing another vertical line from that mark until it hits the parabola. Here a short cut is employed by drawing a horizontal line from point A to the diagonal line $y = x$; I call the new point of intersection B. Note that point B and the origin lie at diagonally opposite corners of a square whose sides have a length equal to the value of y determined in the first iteration. As a consequence the y value can be fed back into the system by drawing a vertical line from B until it hits the parabola (point C). By continuously repeating the procedure of moving vertically until the parabola is hit and moving horizontally until the diagonal line is hit one produces a rectangular path that spirals into a square.



A bifurcation diagram shows the transition to chaos

The geometric recipe mimics the core procedure within CHAOS1. The two places where the resultant square intersects the parabola correspond to the two-point attractor. Enterprising programmers might undertake the interesting project of generating such figures by computer. In doing so armchair investigators could gain insights into the "simple" iterative equation under study. Specifically, what do the figures look like when chaos sets in? Are the random-looking numbers generated by values of r that produce chaos truly random?

I owe the idea for an excursion into chaos to a number of readers who wrote in. Among them was James P. Crutchfield, one of the authors of the article "Chaos" referred to above. Crutchfield and his coauthors explain that "the key to understanding chaotic behavior lies in understanding a simple stretching and folding operation, which takes place in the state space." In the case of the simple amplifier system the state space is a line segment that contains the attractor points and

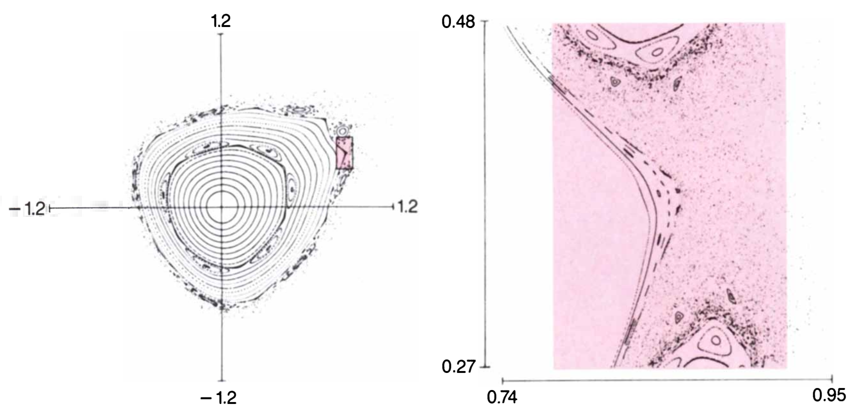


A two-point attractor appears in a geometric simulation of a simple system

the point representing the current value of x. Where do the stretching and folding come in?

Iterating the equation $x \leftarrow rx(1-x)$ amounts to mapping the points between 0 and 1 into a parabolic curve. Points that are close together on the unit interval, particularly those close to 0, end up farther apart when they are mapped into the parabolic curve. This happens, of course, when the number $rx(1-x)$ replaces x. The folding operation comes about because of the bilateral symmetry of the parabola; except at the apex of the curve there are always two points on the unit interval that map into the same value $rx(1-x)$. Those points are of course x and $1-x$.

Much of the structure of the bifurcation diagram has been analyzed by chaos theorists. The boundaries of the chaotic regions are set by the minimum and maximum values of the iterates of $x = .5$. The curves followed by the minima and maxima, as well as those followed by the "veils" that hang so strangely down in the chaotic regions, are all simple polynomials in r. At the places where the shading is densest one finds the highest concentration of points in the strange attractors that cross them. In the empty bands mentioned above chaos gives way to order. Theory tells us that for every whole number there is a band (however narrow) with orbits of precisely that size. Finally, it will come as no surprise to readers familiar with chaos that strange attractors, even in the humble system just explored, have a fractal nature; an infinite number of points show interesting detail at all levels of magnification, like the Mandelbrot set described in this column in August, 1985.



Successive orbits of a Hénon mapping **(left)** *degenerate into chaos* **(right)**

More complicated dynamical systems are embodied in the equations named after Michel Hénon, a French mathematician. The so-called Hénon mappings not only describe physical systems such as moving asteroids and dripping faucets but also generate beautiful images in the process. A Hénon mapping consists of not one equation but two. Here is an example:

$$x \leftarrow x\cos(a) - (y - x^2)\sin(a)$$
$$y \leftarrow x\sin(a) + (y - x^2)\cos(a)$$

Current values of two variables x and y are used in the right-hand sides of both equations to produce new values (also symbolized by x and y) in the left-hand sides.

A program called CHAOS2 exploits the two equations to produce images of the order and chaos inherent in a wide class of dynamical systems. CHAOS2 has a core program that is similar to the core program of CHAOS1:

```
input x and y
for i ← 1 to 1,000
  xx ← xcos(a) − (y − x²)sin(a)
   y ← xsin(a) + (y − x²)cos(a)
   x ← xx
   plot (100x, 100y)
```

The differences between the two core programs stem from two sources: CHAOS2 has two iterated variables instead of one, and the system described by the Hénon mapping is conservative rather than dissipative. The presence of two variables forces one to employ a temporary variable xx for the new value of x while the current value of x is still being used in the second equation. The fact that the underlying dynamical system is conservative means the primary iteration loop for eliminating transient values can be removed. There are no losses of energy due to friction or other dissipative leaks. Consequently there are no attractors as such. One might say, however, that every orbit computed by the system is its own attractor. In any event, strangeness (and chaos) is certainly present in the Hénon mapping. Finally, for each setting of the parameter a the resulting system has a multitude of orbits and, owing to conservatism, any initial pair of values for x and y will represent a point that is already on one of the orbits; the attraction is instant, so to speak. For these

The Hénon mapping generates different figures for **a** = *.264* **(left)** *and* **a** = *1.5732* **(right)**

reasons the core of CHAOS2 does not use standard initial values for its iteration variables. They must be input by the computer programmer.

CHAOS2 is complete when its core is preceded by an input statement that allows the programmer to select the value of a. As in CHAOS1, each new value of a leads to a new system. But because the system is two-dimensional, a sampling of orbital plots takes up all available room; one cannot systematically vary the control parameter a without invoking chaos of an unwanted kind.

The user of CHAOS2 therefore specifies an initial orbit by typing in the coordinates of a point on it. Sitting back, he or she watches in fascination as the orbit is plotted. It might turn out to be a curve (traced not continuously but intermittently) or it might turn out to be something a little stranger. For example, the bottom illustration on the opposite page displays a succession of 38 orbits in a Hénon mapping with the value of a set at 1.111. From the center of the plot outward the orbits form a nest of closed curves until the sudden appearance of small "islands": individual orbits wedged between the larger nested ones. Farther out the nested orbits continue until the onset of chaos. In the outer reaches of the phase plot more islands appear, along with a random sprinkling of points that denotes the onset of chaos. One of the chaotic areas (outlined by a rectangle in the illustration) is shown in magnified form. Readers who want to magnify Hénon diagrams are warned to use the most precise arithmetic available on their machines.

As I have mentioned, Hénon mappings represent a great variety of conservative systems, such as asteroids orbiting the sun. Unfortunately the orbits in the diagrams are not the orbits of the asteroids but phase plots of those orbits. In the diagram just described the horizontal axis may represent the position of an asteroid in terms of its distance from the sun. The vertical axis may represent the radial velocity, or the rate of change, in this distance. Each point on the orbit computed by the Hénon mapping represents the radial distance and velocity of an asteroid at a specific angular position with respect to the sun, that is, when the asteroid passes through a vertical plane making that angle with the sun. Successive points computed by the mapping represent the asteroid's successive reappearances on the plane. The islands mentioned above are resonance bands due to perturbations in the asteroid's orbit by larger bodies in the solar system such as Jupiter. In the chaotic regions the radial position and velocity of an asteroid will vary in an essentially random way every time an asteroid revisits the specified plane. Its motion is unpredictable. Almost anything can happen.

On the aesthetic side it is worth looking at some other plots generated by Hénon mappings; quite apart from physical interpretations, there is a whimsical quality present, as is seen in the illustration above. They resemble strange, aquatic creatures.

Readers wanting to learn more about Hénon mappings should get a copy of the December 1986 issue of *Byte* magazine. There Gordon Hughes, a professor of mathematics at California State University, has engagingly described some of the relevant physics and mathematics underlying Hénon mappings. PASCAL programs are also listed.

A reader in Holland, Peter de Jong of Leiden, has already suggested some other iteration formulas that produce bizarre shapes and images. He recommends the four-parameter iterations $x \leftarrow \sin(ay) - \cos(bx)$ and $y \leftarrow \sin(cx) - \cos(dy)$. Begin with x and y both set equal to 0. Then, to get the figure de Jong calls "chicken legs," try $a = 2.01$, $b = -2.53$, $c = 1.61$ and $d = -.33$. The respective values -2.7, $-.09$, $-.86$ and -2.2 yield a "dot launcher," and the values -2.24, $.43$, $-.65$ and -2.43 produce a "self-decorating Easter egg."

Readers are free, like de Jong, to invent their own iteration formulas and to experiment with them. Anyone finding particularly attractive (or puzzling) chaos is hereby invited to send it to me in care of this magazine. Crutchfield has kindly agreed to correspond directly with those readers whose puzzlement I am not likely to satisfy. He can be reached at the Department of Physics, University of California, Berkeley, Calif. 94720.

Readers with one-track minds have by now undoubtedly solved last month's problem involving the interchange of two cars across a weak bridge. The cars are on a circular track, and an engine occupies another track connected to the circular one by a switch. The bridge is strong enough to hold one car but not the engine. How can the engine switch the cars?

The engine enters the circular track, goes to car A and pushes it onto the bridge. Then the engine backs around the track to car B, couples with it, pushes it to the edge of the bridge and couples B with A. Chugging back to the switch, the engine and two cars back onto the straight track, where A is uncoupled. Next the engine takes B back to the bridge, leaving it uncoupled there. Finally the engine circles the track, pulls B off the bridge to its new position and then retrieves A.

In the April column on computer music I left readers to ponder how to obtain long, nonrepeating sequences of notes by selecting numbers modulo m. The method of selection involved starting with a seed number and from then on continually multiplying by a number a, adding another number b and reducing the result by taking the remainder on division by m. If the numbers a and m are relatively prime (have no common factor larger than 1), the sequence will be the longest one possible. It will also produce the strangest music.

Peter de Jong notes that he has created strange music through chaos. Readers can create similar sounds by converting the numbers generated by CHAOS1 into musical notes. Outside chaotic zones there will be simple, repetitive musical phrases; inside the zones will be the very sounds of chaos.